gr-qc/9410039 UCSBTH-nn-nn

Dynamical Systems Treatment of Scalar Field Cosmologies

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(October , 1994)

Abstract

The conformal equivalence of some cosmological models in Brans-Dicke theory to general relativistic cosmologies with a scalar field is discussed. In the case of radiation-dominated universes, it is shown that the presence of the scalar field has a negligible impact upon the evolution of the models in the Einstein frame. It is also shown that power-law inflation in general relativity, which is conformal to "extended" power-law inflation in Brans-Dicke theory, is not a unique attractor for expanding closed universes, but rather that the occurence of inflation depends upon the initial kinetic energy of the scalar field.

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I. INTRODUCTION

In inflationary universe scenarios, it is usually assumed that general relativity (GR) is the correct theory of gravitation, and the matter is generally taken to be a homogeneous scalar field ϕ, with a potential $V(\phi)$ acting as the vacuum energy to drive the accelerated expansion. The gravitational action and the form of this potential should, in principle, be motivated by the fundamental physics which governs the very early universe, and which would also give rise to the scalar field itself. However, as such a theory is not yet definitely known, it is of interest to study a wide variety of scenarios. In GR, for example, if the potential is simply a constant, $V(\phi) = V_0$, then the spacetime is deSitter and the expansion is exponential. If the potential is exponential, $V(\phi) = V_0 e^{-\lambda\phi}$, then there is a power-law inflationary solution [1]. Models of "extended" inflation [2], on the other hand, choose Brans-Dicke (BD) theory [3] as the correct theory of gravity, and in this case the choice of a constant vacuum energy yields a power-law solution directly [4], whereas exponential expansion may only be obtained if a cosmological constant is explicity inserted into the field equations [5–7], in which case it cannot be interpreted as the vacuum energy of any physical matter field. It is well known that the former case is conformally equivalent to power-law inflation in general relativity [8].

Recently, it was shown that the field equations for cosmology in Brans-Dicke theory may be reduced to a two-dimensional dynamical system for any reasonable perfect fluid matter source [7,9,10]. This is possible for all values of spatial curvature if there is no cosmological constant, and also for spatially flat models with a nonzero cosmological constant. Here we point out that the conformal transformation between the Brans-Dicke frame and the Einstein frame preserves this dynamical system treatment in some cases, and we use this fact in order to make comments about some general relativistic cosmological models, based upon results which have previously been obtained in the Brans-Dicke frame.

The paper is organized as follows. In Sec. II, we discuss the conformal equivalence of Brans-Dicke theory to general relativity with a scalar field. In Sec. III, we summarize our dynamical system formalism for BD cosmologies and state its relevance to the Einstein frame, and in Sec. IV we apply the formalism to the models of interest. Sec. V presents conclusions.

II. THE BRANS-DICKE AND EINSTEIN FRAMES

A. Conformal Equivalence

Including a possible cosmological constant, Brans-Dicke theory may be formulated in terms of the action

$$S_{BD} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi(R-2\Lambda) - \omega\frac{g_{\mu\nu}\phi^{,\mu}\phi^{,\nu}}{\phi} - 16\pi\mathcal{L}_m\right]. \tag{2.1}$$

As Dicke himself pointed out in a companion to his seminal paper with Brans [11], one can reformulate this theory as one in which the Einstein equations are formally satisfied via the conformal transformation

$$\tilde{g}_{\mu\nu} = G\phi g_{\mu\nu}, \tag{2.2}$$

where G is the value Newton's gravitational constant observed today. If one makes the further field redefinition

$$G\phi = \exp\left(\psi\sqrt{\frac{8\pi G}{\omega + 3/2}}\right), \tag{2.3}$$

one arrives at the new action

$$\tilde{S}_{BD} = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left[(\tilde{R} - 2\Lambda) - 16\pi G(\tilde{\mathcal{L}}_m + \frac{1}{2}\tilde{g}_{\mu\nu}\psi^{,\mu}\psi^{,\nu})\right], \tag{2.4}$$

where

$$\tilde{\mathcal{L}}_m = \exp\left(-2\psi\sqrt{\frac{8\pi G}{\omega + 3/2}}\right)\mathcal{L}_m. \tag{2.5}$$

This is the action for general relativity with a massless scalar field which is exponentially coupled to the other matter.

B. Field Equations

The field equations for Brans-Dicke theory, obtained by varying Eq. (2.1), are

$$2\omega\phi^{-1}\Box\phi - \omega\frac{\phi^{,\mu}\phi_{,\mu}}{\phi^2} + R - 2\Lambda = 0 \tag{2.6}$$

and

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 8\pi\phi^{-1}T_{\mu\nu} + \left(\frac{\omega}{\phi^2}\right)(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi^{,\rho}\phi_{,\rho}) + \phi^{-1}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi), \tag{2.7}$$

where

$$\Box\phi \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\mu}\left(\sqrt{-g}\frac{\partial\phi}{\partial x^\mu}\right). \tag{2.8}$$

Taking the trace of Eq. (2.7) and combining the result with Eq. (2.6), one finds

$$\Box\phi = \left(\frac{8\pi}{3 + 2\omega}\right)\left(T^\mu{}_\mu - \frac{\Lambda\phi}{4\pi}\right). \tag{2.9}$$

One generally takes Eqs. (2.7) and (2.9) as the independent field equations.

In the Einstein frame, one of course recovers the usual Einstein equations,

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} + \tilde{g}_{\mu\nu}\Lambda = 8\pi\tilde{T}_{\mu\nu}, \tag{2.10}$$

where now

$$\tilde{T}_{\mu\nu} = \tilde{T}^{\psi}_{\mu\nu} + \tilde{T}^{m}_{\mu\nu}$$
$$= \psi_{,\mu}\psi_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}_{\rho\sigma}\psi^{,\rho}\psi^{,\sigma} + \tilde{T}^{m}_{\mu\nu}. \tag{2.11}$$

It then follows from the Bianchi identities that $\tilde{T}^{\mu\nu}{}_{;\nu} = 0$, from which one sees that the ordinary matter will obey the proper conservation equation, $\left(\tilde{T}^{\mu\nu}{}_{;\nu}\right)^{m} = 0$, only if the corresponding equation of motion for the scalar field is satisfied, *i.e.,* only if $\left(\tilde{T}^{\mu\nu}{}_{;\nu}\right)^{\psi} = 0$, which implies that $\tilde{\Box}\psi = 0$.

Under the double transformation given by Eqs. (2.2) and (2.3), one finds that Eq. (2.9) becomes

$$e^{2c_1\psi}\tilde{\Box}\psi = \frac{c_1}{2}\left[T^{\mu}{}_{\mu} - \frac{\Lambda}{4\pi G}e^{c_1\psi}\right], \tag{2.12}$$

where $c_1 = \sqrt{8\pi G/(\omega + 3/2)}$. Hence we see that two conditions must be met if results from the Brans-Dicke frame are to be applied to realistic Einstein universes with a scalar field: the stress-energy of the ordinary matter (neglecting the scalar field) must be traceless, and the cosmological constant must vanish. Cases of interest which satisfy these conditions are true-vacuum or radiation-filled universes, and power-law inflationary universes.

III. DYNAMICAL SYSTEM FORMALISM

In [9], it was shown that the field equations for perfect fluid Brans-Dicke cosmologies with vanishing cosmological constant may be reduced to a planar dynamical system, and this system was analyzed qualitatively for several cases of interest. Here we will simply outline the method. First we switch to conformal time $d\tau = dt/a(t)$, and define the new variables

$$\beta \equiv \left(\frac{a'}{a} + \frac{\phi'}{2\phi}\right), \tag{3.1}$$

$$\sigma \equiv A\frac{\phi'}{\phi}, \tag{3.2}$$

where primes represent derivatives with respect to conformal time, and

$$A \equiv \left(\frac{2\omega + 3}{12}\right). \tag{3.3}$$

Next, we parametrize the equation of state by writing

$$p = (\gamma - 1)\rho. \tag{3.4}$$

Writing out Eqs. (2.7) and (2.9) for a FRW metric, assuming that the matter satisfies the usual conservation equation, and using Eqs. (3.1)–(3.4), one can derive the dynamical system

$$\sigma' = (1 - 3\gamma/4)(\beta^2 + k - \sigma^2/A) - 2\beta\sigma, \tag{3.5}$$
$$\beta' = (1 - 3\gamma/2)(\beta^2 + k) - (3 - 3\gamma/2)\sigma^2/A. \tag{3.6}$$

In the special case of the true vacuum ($\rho = 0$), the system reduces to

$$\sigma' = -2\beta\sigma, \tag{3.7}$$

$$\beta' = -2\sigma^2/A, \tag{3.8}$$

which are just Eqs. (3.5) and (3.6) subject to the "vacuum constraint" $\beta^2 + k - \sigma^2/A = 0$ (equivalent to the zeroth component of Eq. (2.7) with $\rho = 0$). These systems may now be analyzed using the standard methods of dynamical systems theory [12].

One can easily verify that under the transformation to the Einstein frame specified by Eqs. (2.2) and (2.3), the dynamical system (3.5, 3.6) is unaltered, provided that we make the additional redefinitions $\beta = b'/b$ and $\sigma = Ac_1\psi'$, where b is the metric in the Einstein frame and the constants A and c_1 were defined in Eqs. (2.12) and (3.3). Note, however, that there will be non-geodesic motion of the cosmological fluid embedded in these equations unless the stress-energy of the ordinary matter is traceless, for reasons which were explained in Sec. II.

IV. RESULTS

Although one can obtain the significant features of the solutions to Eqs. (3.5) and (3.6) analytically, a useful technique is to numerically integrate the system, and then to plot the solution curves (trajectories) in the β–σ plane. Figs. 1 and 2 show the results of this procedure for universes dominated by radiation and false-vacuum energy, respectively. The shaded regions require negative energy density and so are disallowed on physical grounds. The boundaries of these regions represent true vacuum solutions; they are just those trajectories which satisfy the "vacuum constraint" mentioned previously. In the case of $k = 0$, these solutions were found analytically by O'Hanlon and Tupper [13]. The dark solid line represents the static condition $da/dt = 0$ in the Brans-Dicke frame; the dark dashed line represents the analagous condition $db/dt = 0$ in the Einstein frame. Note that, in general, there is a family of solutions for any matter source and spatial curvature, with the models parametrized by the value of σ at a given β. Also note that when $\omega < 0$, nonsingular (for $k = 0, -1$) and perpetually expanding (for $k = +1$) radiation models are possible in the Brans-Dicke frame, due to the domination of the scalar field in the dynamics. The presence of these models corresponds to the rotation of the line $da/dt = 0$ into the physical regime of the β–σ plane. In the Einstein frame, where the scalar is not coupled directly to the metric, this behavior is not possible unless the matter is inflationary, *i.e.,* unless the strong energy condition $p > -\rho/3$ is violated, consistent with the fact that such models are precluded in GR by singularity theorems [14].

Now one can read the possible behavior of the models in either conformal frame directly from the figures. Fig. 1a represents the spatially flat, radiation-filled FRW models. One sees that all of the initially expanding models approach the equilibrium point $(\beta_0, \sigma_0) = (0, 0)$, which represents the usual quasi-static endpoint of a flat-space FRW universe. In the BD frame, we have chosen $\omega = -1/2$, so that nonsingular models which pass smoothly from contraction to expansion are present. This possibility does not exist in the Einstein frame, where all of the contracting models continue to contract until a singularity is reached. In fact, as must be the case, there is no ω–dependence at all among the radiation models in the Einstein frame, as one can see by substituting $\gamma = 4/3$ and $\sigma = Ac_1\psi' \sim \psi'\sqrt{2\omega+3}$ into Eqs. (3.5) and (3.6). Negative curvature models are represented in Fig. 1b, and the results are identical to those for flat-space models, except that the late-time attractor for the expanding models represents linear expansion, as one would expect for open universes. If the curvature is positive as represented by Fig. 1c, then in the Einstein frame all of the models recollapse as one would expect, although in the BD frame perpetual expansion is possible for certain initial conditions. One concludes that for the radiation-dominated universes, the presence of the scalar field has a negligible effect upon the dynamics of the models in the Einstein frame.

On the other hand, consider a constant vacuum energy in the BD frame, such as would be supplied by an inflaton field which was trapped in a false minimum of its potential, and which produces power-law inflation in both conformal frames. In the Einstein frame, one is considering a scalar field with exponential potential $V(\psi) \sim e^{-\lambda\psi}$, where λ is related to ω by $\lambda = [32\pi G/(\omega+3/2)]^{1/2} = 2c_1$ [8]. For many expanding models there is an attractor, which is the power-law solution first found by Mathiazhagan and Johri [4], and later used by La and Steinhardt in their original model of extended inflation [2]. The solution is $a(t) \sim t^{\omega+1/2}$ in the BD frame; in the Einstein frame it is $b(t) \sim t^{\omega/2+3/4}$. Clearly the value $\omega = 1/2$ seperates models which can inflate from those which cannot. Consider then only those models which fall in the inflationary part of the parameter space, *i.e.,* consider $\omega > 1/2$ or, equivalently, $\lambda < (16\pi G)^{1/2}$. In Figs. 2a and 2b, we have chosen $\omega = 10.5$, and one sees that the zero- and negative-curvature models all inflate eventually. However, it is apparent from Fig. 2c that models with positive curvature do not inflate unconditionally; rather, for some initial conditions recollapse occurs. The same statement therefore holds in both conformal frames: if the rate of change of the scalar field is too high relative to the expansion rate of the universe, recollapse rather than inflation will occur. In the case of power-law inflation in GR, this was first shown by Halliwell using a slightly different formalism [15].

V. CONCLUSIONS

We have seen that the conformal equivalence of BD theory to GR with a scalar field preserves a dynamical systems treatment of cosmological models in some cases of interest, including radiation-filled universes and power-law inflationary models. In the case of radiation, the presence of the scalar field has a negligible impact upon the late-time behavior of the models. This is not surprising, as the matter in this case is completely decoupled from scalar field in both conformal frames. In the case of power-law inflation, one finds that if the scalar field dominates the dynamics, some positively-curved models will recollapse rather than inflate. Hence one cannot assume, either in extended inflationary scenarios in Brans-Dicke theory, or in power-law inflationary scenarios in general relativity, that accelerated expansion will actually occur. In particular, positively curved universes where the energy of the scalar field is high relative to the expansion rate may not inflate. Hence the initial conditions of the inflationary epoch must be taken into consideration. One can speculate that similar statements may hold in other inflationary scenarios, such as "hyperextended inflation" in Brans-Dicke theory [16], or models in GR where the scalar field has a polynomial potential.

VI. ACKNOWLEDGEMENTS

S.K. was supported in part by the National Science Foundation under Grants No. PHY89-04035 and PHY90-08502. B.H. was supported by a SURF fellowship from the College of Creative Studies at UC Santa Barbara.

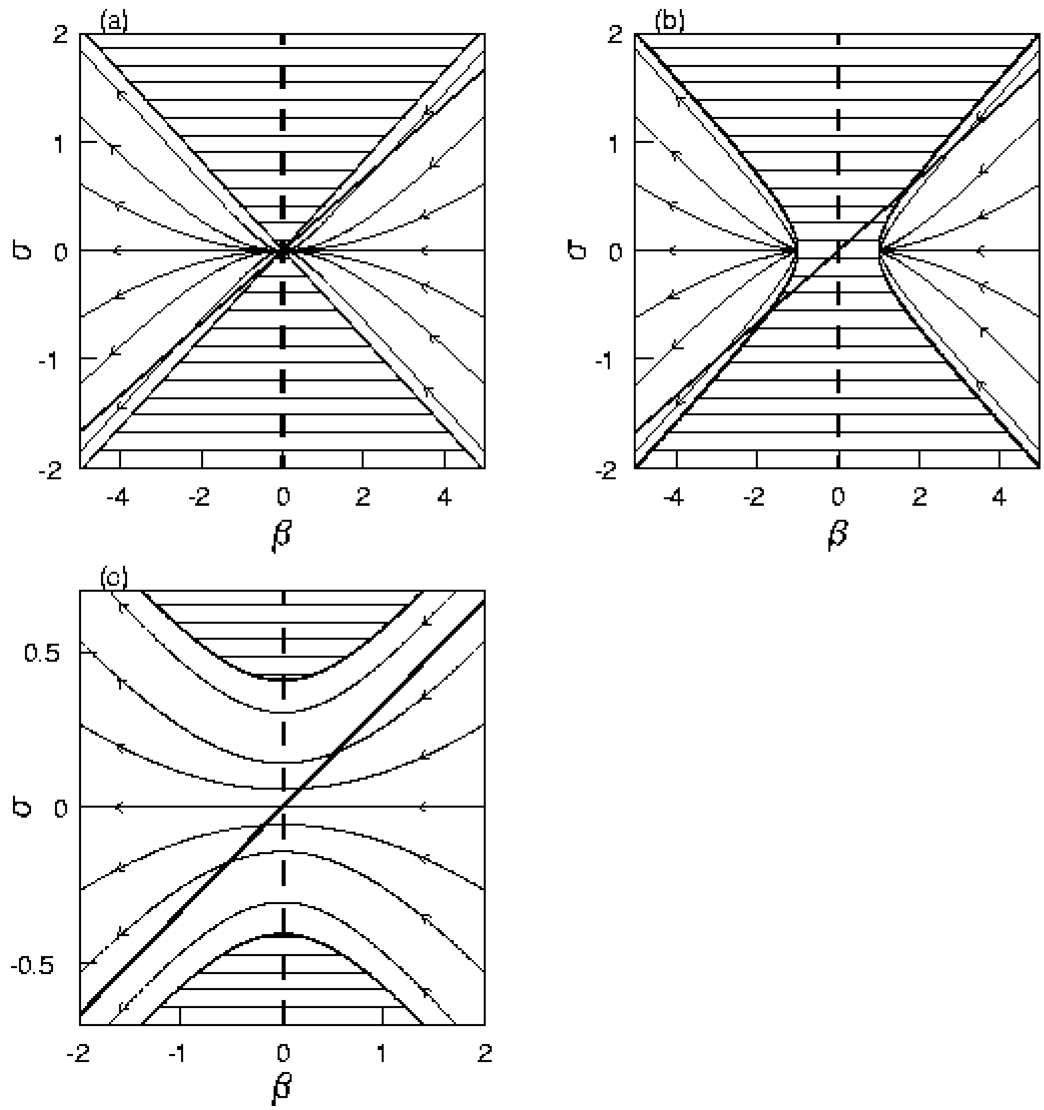
REFERENCES

[1] F. Lucchin and S. Matarrese, *Phys. Rev. D* **32**, 1316 (1985).
[2] D.A. La and P.J. Steinhardt, *Phys. Rev. Lett.* **62**, 376 (1989).
[3] C. Brans and R.H. Dicke, *Phys. Rev.* **124**, 925 (1961).
[4] C. Mathiazhagan and V.B. Johri, *Class. Quantum Grav.* **1**, L29 (1984).
[5] C. Romero and A. Barros, *Gen. Rel. and Grav.* **23**, 491 (1993).
[6] J.D. Barrow and K. Maeda, *Nucl. Phys.* **B341**, 294 (1990).
[7] S. Kolitch, submitted to *Classical and Quantum Gravity*, gr-qc preprint 9409002.
[8] See, for example, B. Campbell, A. Linde and K. Olive, *Nucl. Phys.* **B355**, 146 (1991).
[9] S. Kolitch and D. Eardley, to be published in *Annals of Physics*, gr-qc preprint 9405016.
[10] D. Wands, Unpublished Ph.D. Thesis, University of Sussex, 1993.
[11] R.H. Dicke, *Phys. Rev.* **125**, 2163 (1962).
[12] For a pedagogical review of the stability analysis of dynamical systems, see, for example, M.W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra* (Academic Press, New York, 1974).
[13] J. O'Hanlon and B.O.J. Tupper, *Nuovo Cimento* **7B**, 305 (1972).
[14] S.W. Hawking and R. Penrose, *Proc. R. Soc. London* **A314**, 529 (1969).
[15] J.J. Haliwell, *Phys. Lett.* **185B**, 341 (1987).
[16] P.J. Steinhardt and F.S. Accetta, *Phys. Rev. Lett.* **64**, 2740 (1990)

FIGURES

FIG. 1. The evolution of radiation-filled universes in Brans-Dicke theory with $\omega = -1/2$, and in general relativity with a massless scalar field. The shaded regions require $\rho < 0$ and so are disallowed physically. The dark solid line signifies $da/dt = 0$ in the BD frame; the dark dashed line signifies $db/dt = 0$ in the Einstein frame. In each frame the trajectories to the right of the static line represent expanding models, and those to its left represent contracting models. Hence nonsingular universes are possible only in the BD frame. (a) Models with vanishing spatial curvature ($k = 0$). All universes which start with a big bang asymptotically approach quasistatic equilibrium at $(\beta_0, \sigma_0) = (0, 0)$. (b) Negative-curvature models ($k = -1$). The endpoint for expanding models now represents linear expansion. (c) Positive-curvature models. All models recollapse in the Einstein frame, although some expand perpetually in the BD frame.

FIG. 2. Models dominated by false-vacuum energy in the BD frame with $\omega = 10.5$, corresponding to an exponential potential for the scalar field in the Einstein frame with $\lambda = (8\pi G/3)^{1/2}$. (a) and (b) represent flat-space and negative-curvature models, respectively, all of which approach power-law inflation at late times. (c) Positive-curvature models. Some models are possible which start from a big bang, but then recollapse rather than inflate, due to the domination of the scalar field in the dynamics.



Kolitch and Hall, "Dynamical Systems Treatment of Scalar Field Cosmologies"
Figure 1

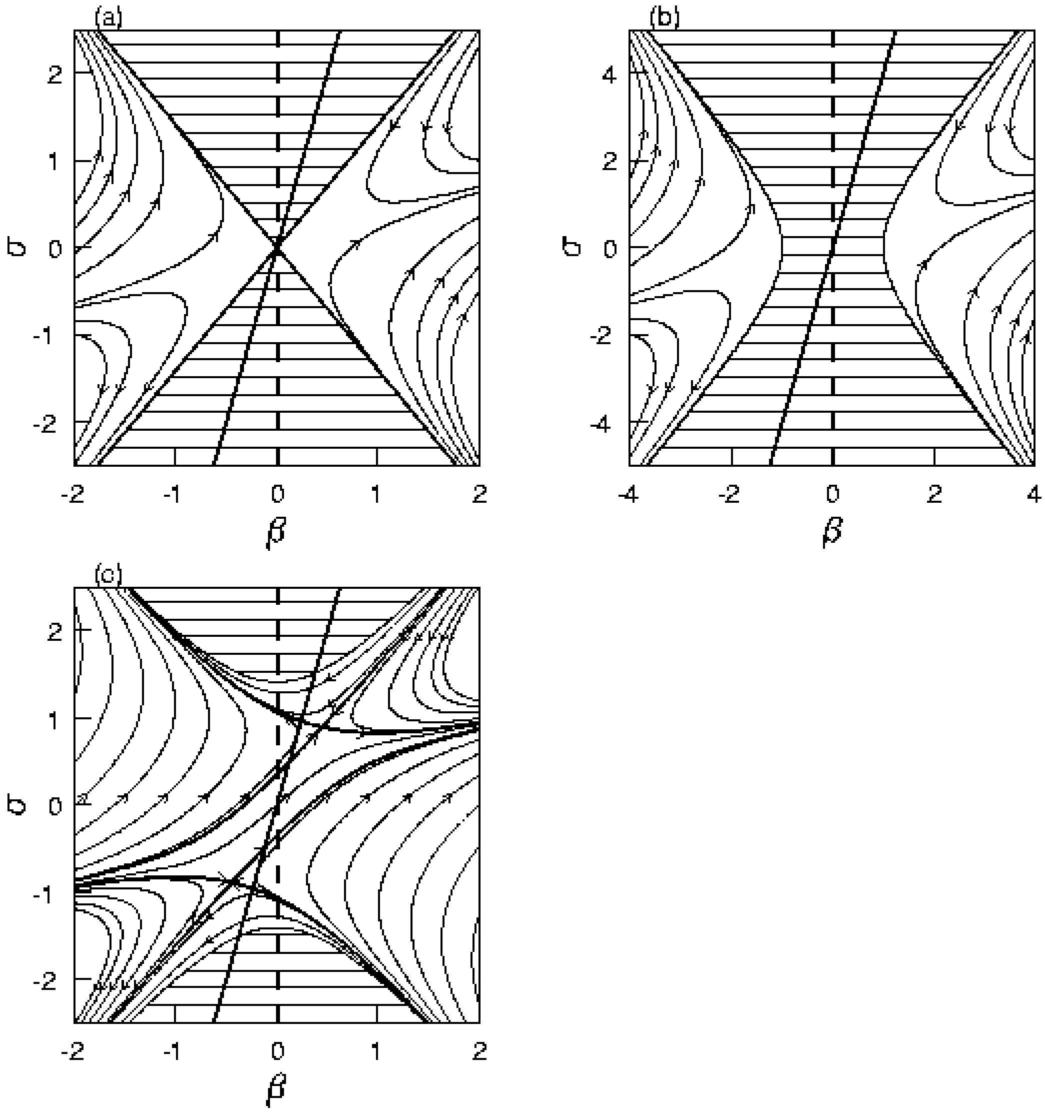


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Figure 2