

Quantum Decay of Domain Walls in Cosmology II: Hamiltonian Approach

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Abstract

This paper studies the decay of a large, closed domain wall in a closed universe. Such walls can form in the presence of a broken, discrete symmetry. We study a novel process of quantum decay for such a wall, in which the vacuum fluctuates from one discrete state to another throughout one half of the universe, so that the wall decays into pure field energy. Equivalently, the fluctuation can be thought of as the nucleation of a second closed domain wall of zero size, followed by its growth by quantum tunnelling and its collision with the first wall, annihilating both. We therefore study the 2-wall system coupled to a spherically symmetric gravitational field. We derive a simple form of the 2-wall action, use Dirac quantization, obtain the 2-wall wave function for annihilation, find from it the barrier factor for this quantum tunneling, and thereby get the decay probability. This is the second paper of a series.

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I. INTRODUCTION

It is well appreciated that domain walls can wreck a cosmological model, and therefore it is of interest to find processes that can destroy domain walls in the early universe. In paper I [1] we introduced a novel process, the quantum decay of domain walls by global fluctuation and quantum tunnelling. We studied a closed universe dominated by a single closed domain wall — the Vilenkin-IPser-Sikivie or “VIS” solution [2,3] — and we found an instanton that mediates its decay into a closed universe containing pure field energy. However some technical problems cropped up in the instanton calculation. Therefore, in this paper we will study the same decay process by a different technique, namely a Hamiltonian formulation and Dirac quantization.

We must first explain why gravity is involved in this decay at all. A domain wall in flat spacetime separates two infinite regions of different discrete vacuum state, in the presence of a broken discrete symmetry. The wall cannot decay because any quantum fluctuation into a no-wall state has an infinite barrier. The Vilenkin solution [2] seems to describe an infinite domain wall dressed by its gravitational field; however this spacetime is not geodesically complete, and its complete analytic continuation [3] can be interpreted as a closed, topologically S^3 universe dominated by a closed finite S^2 domain wall. This universe (we call it the VIS solution) starts at infinite volume, collapses to a minimum volume, at which point it halts and then re-expands to infinite volume. The minimum radius of the domain wall is $R_{\min} \sim 1/\sigma G$ where σ is wall surface tension, and so gravity helps set this scale R_{\min} . This is the archetype of a universe dominated by a domain wall, and the domain wall is classically forbidden from collapsing to zero radius. However, the universe is of finite volume, $\sim R_{\min}^3$ near minimum, so the wall is subject to decay by global quantum fluctuations, in which the vacuum state in one whole half of the universe jumps to the same state as the other half. Clearly this decay process has a finite, albeit large, barrier factor $\sim \sigma^{-2} G^{-3}$. Thus, the domain wall decay problem becomes a problem in quantum gravity.

The decay process can more particularly be regarded as follows. A second closed domain wall nucleates at zero size in the original universe, and the two walls then approach each other by quantum tunnelling. When the two walls meet, they annihilate into pure field energy. Figure 1 illustrates the 1-wall VIS universe itself, and also the 2-wall decay process. For this reason we study the spherically symmetric 2-wall system, coupled to a gravitational field, in this paper. An important technical ingredient in this study is the result of Thiemann and Kastrup [4], who found an elegant pair of canonical variables (T, M) for spherically symmetric gravitational field configurations. Here we also find compatible canonical variables for domain walls.

Quantum tunneling of domains is already well known in condensed matter physics, and has been studied both theoretically and experimentally; see *e.g.*, [5–8]. This gives hope that similar processes can be understood in cosmology.

Section II is devoted to deriving the first main technical result of this paper, a simple form of the effective action for the 2-wall system, obtained by integrating out the spherically symmetric gravitational field:

$$\begin{aligned}
S = \int dt & \left\{ iR_1^2 \dot{\psi}_1 + iR_2^2 \dot{\psi}_2 + MT \right. \\
& - \tilde{N}_1^t \left[\mu^2 R_1^2 - 4\sqrt{1 - 2M/R_1} \sin^2 \psi_1 - \left(1 - \sqrt{1 - 2M/R_1} \right)^2 \right] \\
& \left. - \tilde{N}_2^t \left[\mu^2 R_2^2 - 4\sqrt{1 - 2M/R_2} \sin^2 \psi_2 - \left(1 - \sqrt{1 - 2M/R_2} \right)^2 \right] \right\}.
\end{aligned}$$

Here R_1, R_2 are the radii of the two walls (defined by Area= $4\pi R^2$), ψ_1 and ψ_2 are certain (imaginary) time coordinates along the world sheets of the two walls, (T, M) are the Thiemann-Kastrup variables for the region between the two walls, and $\mu \equiv 4\pi\sigma$. Quantization of this action is straightforward and is carried out in Sect. III; then boundary conditions are set, and the second main result is then derived, the quantum tunnelling probability for domain wall decay,

$$P \sim \exp\left(-2\pi/\mu^2 G^3\right). \quad (1)$$

Section IV presents a slightly different quantization, in which interpretation is a bit clearer. Section V discusses the results and compares to paper I. In Appendix A we review Dirac's method of Hamiltonian quantization [9–11].

FIGURES

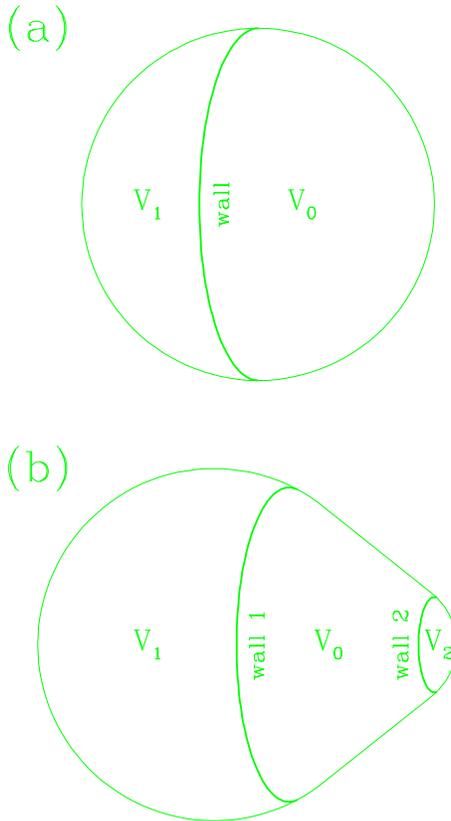


FIG. 1. Spaces dominated by one domain wall and by two domain walls. The spaces are closed and topologically S^3 ; heavy curves denote closed S^2 domain walls. (a) A space slice of the VIS spacetime. Regions V_1 and V_0 are slices of flat spacetime, meeting at the domain wall. (b) A space slice in the quantum tunnelling regime. The second wall has nucleated at zero size and the two walls are tunnelling toward each other; when they meet they will annihilate into pure field energy. Regions V_1 , V_0 and V_2 are slices of flat Euclidean space.

II. CANONICAL ACTION FOR THE TWO WALL SYSTEM

Dirac first applied his method of Hamiltonian quantization (described in Appendix A) to general relativity in [10], and the theory was soon developed in greater detail by Arnowitt, Deser and Misner [12]. It is not the purpose of the present study to review the details of this subject, however, but rather to use the canonical formalism as an alternate approach to the instanton calculation of paper I, which was shown to have a certain pathology. The

reader is referred to the literature for a complete treatment of canonical quantum gravity [13,14].

The quantization of a spherically symmetric spacetime with a finite number of degrees of freedom is a minisuperspace model known as the Berger-Chitre-Moncrief-Nutku (BCMN) model [15–17]. We will be quantizing the spherically symmetric domain-wall spacetime (VIS spacetime [2,3]) introduced previously, while allowing for the possibility that a second domain wall tunnels from zero size and annihilates with the existing one.¹ We therefore expect that there will be just two degrees of freedom in the problem, corresponding to the radii of the two domain walls. A related problem with one degree of freedom, the canonical quantization of a spherical bubble of false-vacuum, was worked out in the WKB approximation by Fischler, Morgan and Polchinski [18]. This same problem was earlier studied using the Euclidean approach by Blau, Guendelman and Guth [19], who found a pathology very similar to what was encountered in the instanton calculation of our previous paper [1].

A. The First Order Action

The action for gravity plus domain walls is given in the thin-wall limit by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) - \frac{\mu}{4\pi} \int_{\text{walls}} d^3A, \quad (2)$$

where again $\mu/4\pi$ is σ , the energy per unit area of a domain wall, and where the spherically symmetric metric as

$$ds^2 = -(N^t dt)^2 + L^2(dr + N^r)^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (3)$$

In what follows, we will take $\Lambda = 0$, corresponding to the case of pure domain walls. Also, except where noted, we work in units where $G = 1$. The canonical coordinates, which are functions of (r, t) , are $(N^t, N^r, L, R, r_1, r_2)$, where r_1 and r_2 are the radial coordinates of the two domain walls. Defining the conjugate momenta as usual, the Hamiltonian form of this action is

$$S = \int dt \left[p_1 \dot{r}_1 + p_2 \dot{r}_2 + \int dr (\pi_L \dot{L} + \pi_R \dot{R} - N^t \mathcal{H}_t - N^r \mathcal{H}_r) \right], \quad (4)$$

where [18]

$$\begin{aligned} \mathcal{H}_t &= \frac{L\pi_L^2}{2R^2} - \frac{\pi_L\pi_R}{R} + \frac{1}{2} \left[\frac{2R}{L} \left(\frac{R'}{L} \right)' + \frac{R'^2}{L} - L \right] \\ &\quad + \sum_{j=1,2} \delta(r_j - r) \left(\frac{p_j^2}{L^2} + \mu^2 R_j^4 \right)^{1/2}, \\ \mathcal{H}_r &= R' \pi_R - L \pi_L' - \sum_{j=1,2} \delta(r_j - r) p_j. \end{aligned} \quad (5)$$

¹One might more generally study non-spherically-symmetric tunnelling configurations. We expect these to be no more probable, and to affect the prefactor but not the exponential barrier factor in our result. However, we have no proof of these expectations.

This action is generally covariant under coordinate transformations of (t, r) .

Since there are no time derivatives of N_t and N_r in the action, the primary constraints are

$$\pi_{N^t} = \pi_{N^r} = 0, \quad (6)$$

which are first-class. However, since the Poisson brackets between these constraints and the full Hamiltonian do not vanish, there are the secondary constraints

$$\mathcal{H}_t = \mathcal{H}_r = 0, \quad (7)$$

which are also first-class, and which generate coordinate transformations of (t, r) . Assuming $R(r)$ to be continuous and $\pi_{L,R}$ to be free of delta-functions at each wall, integration of these secondary constraints across a wall implies the following jump conditions at the surface:

$$\begin{aligned} \Delta\pi_L &= -\frac{p}{L}, \\ \Delta R' &= -\frac{E}{R}, \end{aligned} \quad (8)$$

where $E \equiv (p^2 + \mu^2 L^2 R^4)^{1/2}$ evaluated at the wall.

To implement these constraints in the 1-wall system, Fischler, Morgan and Polchinski [18] followed the Dirac approach (see the Appendix A) to find a wave function satisfying

$$\pi_{N^t} |\Psi\rangle = \pi_{N^r} |\Psi\rangle = 0, \quad (9)$$

$$\mathcal{H}_t |\Psi\rangle = \mathcal{H}_r |\Psi\rangle = 0. \quad (10)$$

The first pair of these relations simply says that the wave function is independent of the lapse and the shift; the second pair will generate the dynamics of the wave function.

As has been mentioned, it is generally true that the first-class constraints are in one-to-one correspondence with the gauge symmetries of the theory; the existence of four such constraints in the present case therefore indicates that there are four gauge degrees of freedom. Two of these correspond to the invariance of the theory under different choices of the lapse and shift functions; we are also free to fix the time slicing and radial parametrization through gauge choices. Whether or not one fixes this part of the gauge before quantization distinguishes Dirac quantization from ADM quantization: Dirac's procedure, involving no gauge fixing, leads in principle to the wave function for all possible time slicings and radial parametrizations, whereas in the ADM procedure one fixes the gauge before quantization and winds up with the wave function only for a given slicing and parametrization. The Dirac procedure is generally more unwieldy than the ADM method; however, one must take care that possible quantum behavior is not ruled out by a premature gauge choice. Indeed, it was shown in [18] that overzealous gauge fixing may lead to the inadvertent exclusion of parts of the quantum dynamics.

We will therefore take a hybrid Dirac-ADM approach. Roughly speaking, we will “integrate out the gravitational field”: We will fix the radial coordinate and take a fixed family of time slices, and then solve the constraint equations in the three vacuum regions separated by the two walls, to reduce the action to an effective action which exclusively involves wall degrees of freedom. Then we will implement Eq. (10) solely at the walls.

(In fact, the gauge fixing is a convenience but not a necessity for this problem. A future paper in this series will present a [nearly] gauge invariant derivation of the effective action for n walls in spherical symmetry.)

B. Gauge Fixing

We first look for a solution to the constraints as follows. Let the radial coordinate take the range $0 \leq r \leq r_3$, and divide the (compact) space into three regions. There are two centers of spherical symmetry, located at $r = 0$ and $r = r_3$, corresponding to the centers of the two spherical domain walls, and in addition there is a middle region between the two walls, where $r_1 \leq r \leq r_2$. The walls themselves are located at $r = r_1$ and $r = r_2$. We will refer to the three regions as region V_1, V_0, V_2 respectively:

$$\begin{aligned} V_1 : & \quad 0 \leq r \leq r_1, \\ V_0 : & \quad r_1 \leq r \leq r_2, \\ V_2 : & \quad r_2 \leq r \leq r_3. \end{aligned} \tag{11}$$

We now fix the radial parametrization everywhere by imposing the coordinate gauge condition

$$L = 1, \tag{12}$$

and then impose the slicing condition

$$\begin{cases} R\pi_R = 2\pi_L, & V_1, \\ R\pi_R = \pi_L, & V_0, \\ R\pi_R = 2\pi_L, & V_2. \end{cases} \tag{13}$$

C. Solution of the Constraints

Combining these conditions with the spatial constraint equation

$$\mathcal{H}_r = R'\pi_R - \pi_L \tag{14}$$

leads to the solutions

$$\begin{aligned} \pi_R &= \begin{cases} 2ik_1R, & V_1, \\ ic_0, & V_0, \\ 2ik_2R, & V_2, \end{cases} \\ \pi_L &= \begin{cases} ik_1R^2, & V_1, \\ ic_0R, & V_0, \\ ik_2R^2, & V_2; \end{cases} \end{aligned} \tag{15}$$

where k_1, c_0, k_2 are constants of integration. Here and below, the constants of integration that appear will become our degrees of freedom, and should all be understood as functions of time in the dynamical problem. The factors of i have been chosen appropriate to the classically forbidden, tunnelling regime.

The Hamiltonian constraint $\mathcal{H}_t = 0$ then becomes

$$0 = 2\mathcal{H}_t = \begin{cases} 3k_1^2R^2 - 1 + 2RR'' + R'^2, & V_1, \\ c_0^2 - 1 + 2RR'' + R'^2, & V_0, \\ 3k_2^2R^2 - 1 + 2RR'' + R'^2, & V_2. \end{cases} \tag{16}$$

These equations have the general solutions

$$r = \int \frac{dR}{\sqrt{1 - k_1^2 R^2}} \quad (17)$$

in region V_1 (assuming regularity at the origin $r = 0$);

$$r - r_0 = \int \frac{dR}{\sqrt{1 - c_0^2 - 2M/R}} \quad (18)$$

in region V_0 , where M and r_0 are further constants of integration; and

$$r - r_3 = \int \frac{dR}{\sqrt{1 - k_2^2 R^2}} \quad (19)$$

in region V_3 (assuming regularity at the anti-origin $r = r_3$) where r_3 is another constant of integration.

In region V_1 , requiring $R(0) = 0$ gives the solution

$$R(r) = \frac{1}{k_1} \sin(k_1 r), \quad 0 \leq r \leq r_1. \quad (20)$$

Similarly, in region V_2 , requiring $R(r_3) = 0$ gives the solution

$$R(r) = \frac{1}{k_2} \sin[k_2(r_3 - r)], \quad r_2 \leq r \leq r_3. \quad (21)$$

In region V_0 , defining

$$c_0 \equiv \sin \theta_0 \quad (22)$$

we leave the solution in implicit form as

$$r - r_0 = \int \frac{dR}{\sqrt{\cos^2 \theta_0 - 2M/R}}, \quad r_1 \leq r \leq r_2. \quad (23)$$

Here the constants r_0 and r_3 are fixed in terms of the other variables by our requirement that $R(r)$ be continuous across each wall. Eqs. (20)–(23) represent a general solution to the Hamiltonian and spatial constraints, parametrized by $(\theta_0, k_1, k_2, r_1, r_2, M)$. It will be convenient in what follows to define

$$\begin{aligned} R_j &\equiv R(r_j) \quad (j = 1, 2), \\ \theta_1 &\equiv k_1 r_1, \\ \theta_2 &\equiv \pi - k_2(r_3 - r_2); \end{aligned} \quad (24)$$

here and throughout, the index $j = 1, 2$ runs over the two walls. We take as our independent parameters the set $(\theta_0, \theta_1, \theta_2, M, r_1, r_2)$.

Given the above solution to the constraints, it follows from Eq. (15) that

$$\pi_L = \begin{cases} iR \sin(k_1 r), & V_1, \\ iR \sin \theta_0, & V_0, \\ iR \sin[k_2(r_3 - r)], & V_2, \end{cases} \quad (25)$$

and

$$\pi_R = \begin{cases} 2i \sin(k_1 r), & V_1, \\ i \sin \theta_0, & V_0, \\ 2i \sin[k_2(r_3 - r)], & V_2. \end{cases} \quad (26)$$

D. Reduction of the Action

We have now found a 6-parameter family of solutions which identically solve the constraints $\mathcal{H}_{t,r} = 0$ everywhere except at the 2 walls, while at the 2 walls we still have 2 canonical momenta p_j , and 4 constraints, the jump conditions (8). Our reduced phase space is now of finite dimension 8, with coordinates $(\theta_0, \theta_1, \theta_2, r_1, r_2, M, p_1, p_2)$. Our system is described as a time dependent point in phase space obeying the constraints.

We can therefore write the total action as

$$\begin{aligned} S &= \int dt \left\{ p_1 \dot{r}_1 + p_2 \dot{r}_2 + \int dr (\pi_L \dot{L} + \pi_R \dot{R} - N^t \mathcal{H}_t - N^r \mathcal{H}_r) \right\} \\ &= \int dt \left\{ \sum_{j=1,2} (p_j \dot{r}_j - N_j^t [E_j + R_j (\Delta R')_j] - N_j^r [p_j + (\Delta \pi_L)_j]) \right. \\ &\quad \left. + \int dr (\pi_L \dot{L} + \pi_R \dot{R}) \right\}. \end{aligned} \quad (27)$$

where E_1 and E_2 , obtained from the definition $E_j \equiv (p_j^2 + \mu^2 R_j^4)^{1/2}$, are now given by

$$\begin{aligned} E_1 &= [\mu^2 R_1^4 - R_1^2 (\sin \theta_1 - \sin \theta_0)^2]^{1/2}, \\ E_2 &= [\mu^2 R_2^4 - R_2^2 (\sin \theta_0 - \sin \theta_2)^2]^{1/2}; \end{aligned} \quad (28)$$

and where, using Eqs. (20,21,20,25),

$$\begin{aligned} (\Delta R')_1 &= \cos \theta_1 - \sqrt{\cos^2 \theta_0 - 2M/R_1}, \\ (\Delta \pi_L)_1 &= -iR_1 (\sin \theta_1 - \sin \theta_0), \\ (\Delta R')_2 &= -\cos \theta_2 + \sqrt{\cos^2 \theta_0 - 2M/R_2}, \\ (\Delta \pi_L)_2 &= -iR_2 (\sin \theta_0 - \sin \theta_2). \end{aligned} \quad (29)$$

Next we calculate the gravitational contribution to the action, *i.e.*,

$$S_G = \int dt dr (\pi_L \dot{L} + \pi_R \dot{R}). \quad (30)$$

In the gauge $L = 1$, we can neglect the first term. The second term is calculated as follows: write Eqs. (20)–(23) in the form

$$R(r) = A\Theta_A + B\Theta_B + C\Theta_C, \quad (31)$$

where A, B, C are just different names for $R(r)$ in the three regions, and

$$\begin{aligned} \Theta_A &= \Theta(r_1 - r), \\ \Theta_B &= \Theta(r - r_1) - \Theta(r - r_2), \\ \Theta_C &= \Theta(r - r_2) \end{aligned} \quad (32)$$

are step functions for the three regions. Similarly, write Eq. (26) as

$$\pi_R = 2ik_1A\Theta_A + i\sin\theta_0\Theta_B + 2ik_2C\Theta_C. \quad (33)$$

Then it follows that

$$\pi_R \dot{R} = ik_1(A^2)^\bullet \Theta_A + i\sin\theta_0 \dot{B} \Theta_B + ik_2(C^2)^\bullet \Theta_C. \quad (34)$$

Integrating by parts and using the fact that

$$\begin{aligned} \dot{\Theta}_A &= \dot{r}_1 \delta(r_1 - r) \\ \dot{\Theta}_B &= -\dot{r}_1 \delta(r - r_1) + \dot{r}_2 \delta(r - r_2) \\ \dot{\Theta}_C &= -\dot{r}_2 \delta(r - r_2), \end{aligned} \quad (35)$$

one finds

$$\begin{aligned} S_G &= \int dt \left\{ \dot{r}_1 (-ik_1 R_1^2 + i\sin\theta_0 R_1) + \dot{r}_2 (ik_2 R_2^2 - i\sin\theta_0 R_2) \right. \\ &\quad \left. + \int dr (-ik_1^2 A^2 \Theta_A - i\cos\theta_0 \dot{\theta}_0 B \Theta_B - ik_2 C^2 \Theta_C) \right\} \\ &= \int dt \left\{ (\Delta\pi_L)_1 \dot{r}_1 + (\Delta\pi_L)_2 \dot{r}_2 + \int dr (-ik_1 A^2 \Theta_A - i\cos\theta_0 \dot{\theta}_0 B \Theta_B - ik_2 C^2 \Theta_C) \right\}. \end{aligned} \quad (36)$$

where $\theta_1 \equiv k_1 r_1$, $\theta_2 \equiv \pi - k_2(r_3 - r_2)$. Combining Eq. (36) with Eq. (27), the action becomes

$$\begin{aligned} S &= \int dt dr (-ik_1 A^2 \Theta_A - i\cos\theta_0 \dot{\theta}_0 B \Theta_B - ik_2 C^2 \Theta_C) \\ &\quad + \int dt \sum_{j=1,2} \left\{ -N_j^t [E_j + R_j (\Delta R')_j] + [\dot{r}_j - N_j^r] \tilde{p}_j \right\}, \end{aligned} \quad (37)$$

where

$$\begin{aligned} \tilde{p}_1 &\equiv p_1 + (\Delta\pi_L)_1 \equiv p_1 - iR_1(\sin\theta_1 - \sin\theta_0), \\ \tilde{p}_2 &\equiv p_2 + (\Delta\pi_L)_2 \equiv p_2 - iR_2(\sin\theta_0 - \sin\theta_2). \end{aligned} \quad (38)$$

Now consider the terms involving A and C in Eq. (37). Performing the radial integration leads to

$$\begin{aligned} &\int dt dr (-ik_1^2 A^2 \Theta_A - ik_2 C^2 \Theta_C) \\ &= \int dt \left[-\frac{ik_1}{2k_1^2} (r_1 - R_1 \cos\theta_1) - \frac{ik_2}{2k_2^2} (r_3 - r_2 + R_2 \cos\theta_2) \right]. \end{aligned} \quad (39)$$

Furthermore, using the definitions $k_1 R_1 = \sin \theta_1$, $\theta_1 = k_1 r_1$ and $k_2 R_2 = \sin \theta_2$, $\theta_2 = \pi - k_2(r_3 - r_2)$, one can show that

$$\begin{aligned}\frac{\dot{k}_1}{k_1^2}(r_1 - R_1 \cos \theta_1) &= R_1^2 \dot{\theta}_1 + \frac{du}{dt}, \\ \frac{\dot{k}_2}{k_2^2}(r_3 - r_2 + R_2 \cos \theta_2) &= -R_2^2 \dot{\theta}_2 + \frac{dv}{dt},\end{aligned}\tag{40}$$

where

$$\begin{aligned}u &= -\frac{R_1^2}{2 \sin^2 \theta_1}(\theta_1 - \sin \theta_1 \cos \theta_1), \\ v &= -\frac{R_2^2}{2 \sin^2 \theta_2}(\pi - \theta_2 + \sin \theta_2 \cos \theta_2).\end{aligned}\tag{41}$$

Hence we can write, dropping the total time derivatives,

$$\int dt dr (-i \dot{k}_1^2 A^2 \Theta_A - i \dot{k}_2 C^2 \Theta_C) = \int dt (-i R_1^2 \dot{\theta}_1 / 2 + i R_2^2 \dot{\theta}_2 / 2).\tag{42}$$

The term involving B can be rewritten as

$$\begin{aligned}\int dt dr (-i \cos \theta_0 \dot{\theta}_0 B \Theta_B) &= \int dt (-i \cos \theta_0 \dot{\theta}_0) \int_1^2 \frac{R dR}{R'} \\ &= \int dt (-i \dot{\theta}_0) \left[F(M, R_1, R_2) + \frac{R_2^2}{2} - \frac{R_1^2}{2} \right],\end{aligned}\tag{43}$$

where

$$F(M, R_1, R_2) \equiv \int_1^2 R dR \left[\frac{\cos \theta_0}{\sqrt{\cos^2 \theta_0 - 2M/R}} - 1 \right].\tag{44}$$

Combining Eqs. (37), (42) and (43), we find the reduced action

$$\begin{aligned}S &= \int dt \left\{ i R_1^2 \dot{\theta}_1 / 2 + i R_2^2 \dot{\theta}_2 / 2 - i \dot{\theta}_0 \left(R_2^2 / 2 - R_1^2 / 2 + F(M, R_1, R_2) \right) \right. \\ &\quad \left. - \sum_{j=1,2} \left\{ N_j^t [E_j + R_j (\Delta R')_j] + [\dot{r}_j - N_j^r] \tilde{p}_j \right\} \right\}.\end{aligned}\tag{45}$$

Consider the last terms inside the sum over walls, the terms proportional to \tilde{p}_j , in Eq. (45). From the spatial constraints, $\tilde{p}_j = 0$ for all time, so these terms make no contribution to the equations of motion, and can be dropped from the action. The old, gauge-dependent canonical wall coordinates r_j therefore disappear from the action, in favor of the gauge invariant wall quantities $R_j \equiv R(r_j)$. Moreover, the old, gauge-dependent wall canonical momenta p_j have completely decoupled from the remainder of the action, of their own accord, and can be dropped henceforth, along with the spatial constraints. The quantities $i\theta_1$, $i\theta_2$ now act as canonical coordinates for the two walls, and the quantities $R_j^2/2 \equiv R^2(r_j)/2$ now act as gauge-invariant canonically conjugate momenta for the two walls. The quantity $i\theta_0$ is

a single remaining canonical coordinate of the gravitational field in region V_0 , and M is some kind of momentum belonging to it, though not canonically conjugate. Evidently, spherically symmetric gravity is one of those simple gauge theories wherein the unphysical degrees of freedom decouple of their own accord when appropriate canonical coordinates are chosen.

The Hamiltonian constraints at the walls can also be simplified. They now read

$$\sum_{j=1,2} \left\{ -N_j^t [E_j + R_j(\Delta R')_j] \right\} = \quad (46)$$

$$\begin{aligned} & -N_1^t \left\{ \left[\mu^2 R_1^4 - R_1^2 (\sin \theta_1 - \sin \theta_0)^2 \right]^{1/2} + R_1 (\sqrt{\cos^2 \theta_0 - 2M/R_1} - \cos \theta_1) \right\} \\ & -N_2^t \left\{ \left[\mu^2 R_2^4 - R_2^2 (\sin \theta_0 - \sin \theta_2)^2 \right]^{1/2} + R_2 (\cos \theta_2 - \sqrt{\cos^2 \theta_0 - 2M/R_2}) \right\}. \end{aligned} \quad (47)$$

We have here the usual awkwardness that the E_j from Eqs. (28) contain a square root, and in order to obtain a simpler quantum mechanical system, we take the usual remedy and “square out” these constraints as follows. Defining

$$N_j^t \equiv R_j^{-2} \tilde{N}_j^t [E_j - R_j(\Delta R')_j] \quad (j = 1, 2) \quad (48)$$

the constraints become

$$\begin{aligned} & \sum_{j=1,2} \left\{ -\tilde{N}_j^t [E_j^2/R_j^2(\Delta R')_j^2] \right\} = \\ & -\tilde{N}_1^t \left[\mu^2 R_1^2 - 1 + 2(\cos \theta_1 \sqrt{\cos^2 \theta_0 - 2M/R_0} + \sin \theta_1 \sin \theta_0) - 1 + 2M/R_1 \right] \\ & -\tilde{N}_2^t \left[\mu^2 R_2^2 - 1 + 2(\cos \theta_2 \sqrt{\cos^2 \theta_0 - 2M/R_0} + \sin \theta_2 \sin \theta_0) - 1 + 2M/R_2 \right]. \end{aligned} \quad (49)$$

The extra factors introduced into the constraint by Eq. (48) never vanish in the classical regime, and make no difference to the classical equations of motion. Moreover they never vanish in the quantum tunnelling regime and so at most affect the prefactor in the tunnelling calculation. The effective action is now

$$\begin{aligned} S = \int dt \left\{ & iR_1^2 \dot{\theta}_1/2 + iR_2^2 \dot{\theta}_2/2 - i\dot{\theta}_0 \left(R_2^2/2 - -R_1^2/2 + F(M, R_1, R_2) \right) \right. \\ & -\tilde{N}_1^t \left[\mu^2 R_1^2 - 1 + 2(\cos \theta_1 \sqrt{\cos^2 \theta_0 - 2M/R_0} + \sin \theta_1 \sin \theta_0) - 1 + 2M/R_1 \right] \\ & \left. -\tilde{N}_2^t \left[\mu^2 R_2^2 - 1 + 2(\cos \theta_2 \sqrt{\cos^2 \theta_0 - 2M/R_0} + \sin \theta_2 \sin \theta_0) - 1 + 2M/R_2 \right] \right\}. \end{aligned} \quad (50)$$

and the reduced phase space is now 6-dimensional with coordinates $(\theta_0, \theta_1, \theta_2, M, R_1, R_2)$.

E. A Cyclic Time Coordinate

We now wish to find a further coordinate transformation in phase space to canonical form. In Eq. (50), the quantity M which appears in the integral expression for $F(R_1, R_2)$ is the Schwarzschild mass of the region of spacetime between the two walls. Hence we expect, and confirm, that variations of this action lead to $\dot{M} = 0$ as the equation of motion for M .

However, it is clear that there is no cyclic coordinate present for which M is the conjugate momentum. M is *part* of the conjugate momentum of θ_0 , but θ_0 is manifestly not cyclic in the action. It is thus desirable to find the coordinate to which M is canonically conjugate, and which would therefore be cyclic in the action.

Recently, Thiemann and Kastrup found [4] that such a canonically conjugate pair of observables can always be found for spherically symmetric field configurations; see also [20–22]. They worked in the Ashtekar approach to canonical quantum gravity, but their result is general. They showed that for the line element

$$ds^2 = -(N^t dt)^2 + L^2(dr + N^r)^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (51)$$

the time variable conjugate to the Schwarzschild mass is given by

$$T = - \int dr (1 - 2M/R)^{-1} \sqrt{(dR/dr)^2 - L^2(1 - 2M/R)}, \quad (52)$$

which in our setting is given by

$$T = -i \sin \theta_0 \int_{R_1}^{R_2} \frac{dR}{\sqrt{\cos^2 \theta_0 - 2M/R}(1 - 2M/R)}. \quad (53)$$

We would like to relate this to the function F defined by Eq. (44). To do so, note that

$$\begin{aligned} \frac{\partial T}{\partial R_{1,2}} &= -i \sin \theta_0 \left[\frac{1}{(1 - \frac{2M}{R_j})(\cos^2 \theta_0 - \frac{2M}{R_j})^{1/2}} \right]_{R_1}^{R_2}, \\ \frac{\partial T}{\partial \theta_0} &= -i \cos \theta_0 \int \frac{dR}{(\cos^2 \theta_0 - \frac{2M}{R})^{3/2}}, \\ \frac{\partial T}{\partial M} &= -i \sin \theta_0 \left[\int \frac{dR/R}{(1 - \frac{2M}{R})(\cos^2 \theta_0 - \frac{2M}{R})^{3/2}} \right. \\ &\quad \left. + 2 \int \frac{dR/R}{(1 - \frac{2M}{R})^2(\cos^2 \theta_0 - \frac{2M}{R})^{1/2}} \right]. \end{aligned} \quad (54)$$

We can integrate by parts in the first term in $\partial T/\partial M$, after writing the integrand as $[R/(1 - 2M/R)] \cdot [dR/[R^2(\cos^2 \theta_0 - 2M/R)^{3/2}]$ to give

$$\frac{\partial T}{\partial M} = \left[\frac{i \sin \theta_0 R_j}{M(1 - \frac{2M}{R_j})(\cos^2 \theta_0 - \frac{2M}{R_j})^{1/2}} \right]_{R_1}^{R_2} + \frac{T}{M}. \quad (55)$$

The time derivative of T is then given by

$$\dot{T} = \frac{\partial T}{\partial R_1} \dot{R}_1 + \frac{\partial T}{\partial R_2} \dot{R}_2 + \frac{\partial T}{\partial \theta_0} \dot{\theta}_0 + \frac{\partial T}{\partial M} \dot{M}. \quad (56)$$

Furthermore, we can integrate F by parts to find

$$\begin{aligned}
-i\dot{\theta}_0 F &= -i\dot{\theta}_0 \cos \theta_0 \int \frac{RdR}{(\cos^2 \theta_0 - \frac{2M}{R})^{1/2}} + \left[\frac{i}{2} R_j^2 \dot{\theta}_0 \right]_{R_1}^{R_2} \\
&= -\frac{i\dot{\theta}_0 \cos \theta_0 M}{2} \int \frac{dR}{(\cos^2 \theta_0 - \frac{2M}{R})^{3/2}} \\
&\quad - \left[\frac{i\dot{\theta}_0 \cos \theta_0 R_j^2}{2(\cos^2 \theta_0 - \frac{2M}{R_j})^{1/2}} \right]_{R_1}^{R_2} + \left[\frac{i}{2} R_j^2 \dot{\theta}_0 \right]_{R_1}^{R_2}. \tag{57}
\end{aligned}$$

Combining Eqs. (54)–(57), we find that T and F are related by

$$\begin{aligned}
\frac{1}{2}(M\dot{T} - \dot{M}T) &= -i\dot{\theta}_0 F(R_1, R_2, \cos \theta_0, M) - \left[\frac{i}{2} R_j^2 \dot{\theta}_0 \right]_{R_1}^{R_2} \\
&+ \left[\frac{i}{2\sqrt{\cos^2 \theta_0 - 2M/R_j}} \left(R_j^2 \cos \theta_0 \dot{\theta}_0 + \sin \theta_0 \frac{\dot{M}R_j - M\dot{R}_j}{1 - 2M/R_j} \right) \right]_{R_1}^{R_2}. \tag{58}
\end{aligned}$$

Up to a total time derivative, therefore, we just get the desired term $M\dot{T}$ in the action, plus some messy terms at the two walls.

F. Canonical Wall Coordinates

Our next task is to find compatible canonical coordinates (ψ_j, R_j) for each wall. Consider the N_1^t constraint at wall 1. The angles θ_0 and θ_1 both enter into this constraint, but we would like to reduce this to a dependence upon a single angle ψ_1 . Guessing the answer, we can do this by defining ψ_1 by

$$\sin(\theta_1 + 2\psi_1) = \sin \theta_0 / \sqrt{1 - 2M/R_1} \tag{59}$$

which implies

$$\cos(\theta_1 + 2\psi_1) = \sqrt{\cos^2 \theta_0 - 2M/R_1} / \sqrt{1 - 2M/R_1} \tag{60}$$

Then the constraint becomes

$$R_1^2 = 4\sqrt{1 - 2M/R_1} \sin^2 \psi_1 + (1 - \sqrt{1 - 2M/R_1})^2, \tag{61}$$

θ_0 and θ_1 themselves having disappeared as desired. Also, by time-differentiating the definition of ψ_1 we find

$$\dot{\theta}_1 + 2\dot{\psi}_1 = \frac{1}{\sqrt{\cos^2 \theta_0 - 2M/R_1}} \left(\cos \theta_0 \dot{\theta}_0 + \sin \theta_0 \frac{\dot{M}R_1 - M\dot{R}_1}{R_1^2(1 - 2M/R_1)} \right) \tag{62}$$

an expression which furnishes exactly the wall terms that are needed in Eq. (58) for \dot{T} .

Similar equations hold at wall 2:

$$\sin(\theta_2 - 2\psi_2) = \sin \theta_0 / \sqrt{1 - 2M/R_2}, \tag{63}$$

which implies

$$\cos(\theta_2 - 2\psi_2) = \sqrt{\cos^2 \theta_0 - 2M/R_2} / \sqrt{1 - 2M/R_2}. \quad (64)$$

The wall 2 constraint is

$$R_2^2 = 4\sqrt{1 - 2M/R_2} \sin^2 \psi_2 + (1 - \sqrt{1 - 2M/R_2})^2, \quad (65)$$

and for the time derivative

$$\dot{\theta}_2 - 2\dot{\psi}_2 = \frac{1}{\sqrt{\cos^2 \theta_0 - 2M/R_2}} \left(\cos \theta_0 \dot{\theta}_0 + \sin \theta_0 \frac{\dot{M}R_2 - M\dot{R}_2}{R_2^2(1 - 2M/R_2)} \right) \quad (66)$$

G. The Effective Action for Two Domain Walls

Putting all this into our previous action gives

$$S = \int dt \left\{ iR_1^2 \dot{\psi}_1 + iR_2^2 \dot{\psi}_2 + M\dot{T} \right. \\ \left. - \tilde{N}_1^t \left[\mu^2 R_1^2 - 4\sqrt{1 - 2M/R_1} \sin^2 \psi_1 - \left(1 - \sqrt{1 - 2M/R_1} \right)^2 \right] \right. \\ \left. - \tilde{N}_2^t \left[\mu^2 R_2^2 - 4\sqrt{1 - 2M/R_2} \sin^2 \psi_2 - \left(1 - \sqrt{1 - 2M/R_2} \right)^2 \right] \right\}, \quad (67)$$

as the effective action for two domain walls, with canonically conjugate phase space coordinates ² $(T, i\psi_1, i\psi_2, M, R_1^2, R_2^2)$, in which the gravitational field is reduced to the single degree of freedom T , and in which there is one degree of freedom ψ_j belonging to each of the two walls. There remain two constraints.

We note the following simplifications that have taken place along the way in the derivation of this action: 1) The original, non-gauge-invariant, canonical coordinates (r_j, p_j) of the walls have dropped out completely, in favor of effective, gauge-invariant, canonical coordinates $(i\psi_j, R_j)$. 2) The gravitational degrees of freedom have all been integrated out, to leave behind just (T, M) . 3) The gauge dependent coordinates $i\theta_j$ which arose during solution of the constraints have likewise disappeared.

² In this action, the coordinates ψ_j and factors of i have been chosen appropriate to a tunnelling problem. The turning points to the classically allowed regimes of phase space are at $\psi_j = \pm\pi/2, \pm3\pi/2, \pm5\pi/2 \dots$ and, if desired, the coordinates can be analytically continued at those points. See Sect. IV.

III. QUANTIZATION AND THE WAVE FUNCTION OF THE TWO WALL SYSTEM

Starting from the action given by Eq. (67), we now proceed to quantize our 2-wall system, set boundary conditions appropriate to the initial state of the system, solve for the 2-wall wave function, $\Psi(\psi_1, \psi_2, T)$ and so determine the decay amplitude. This system can be quantized exactly and it is unnecessary to resort to the WKB approximation.³ The action, Eq. (67), contains two constraints, which can be treated by Dirac quantization (See Appendix A).

A. Reduction to $M = 0$

In the initial state — a single VIS domain wall at minimum radius — we have $M=0$. Thus the boundary value of the wave function Ψ is independent of T , since (T, M) are canonically conjugate [4,20–22]. But M commutes with the constraints since T is a cyclic coordinate, so Ψ must be independent of T throughout:

$$\Psi(\psi_1, \psi_2, T) = \Psi(\psi_1, \psi_2) \quad (68)$$

Thus the domain wall decay problem reduces to a 4-dimensional phase space in $(\psi_1, \psi_2, R_1, R_2)$ with action

$$S = \int dt \left\{ iR_1^2 \dot{\psi}_1 + iR_2^2 \dot{\psi}_2 - \tilde{N}_1^t [\mu^2 R_1^2 - 4 \sin^2 \psi_1] - \tilde{N}_2^t [\mu^2 R_2^2 - 4 \sin^2 \psi_2] \right\} \quad (69)$$

The canonical coordinates are now $(i\psi_1, i\psi_2)$ and the canonical momenta are

$$\begin{aligned} \pi_1 &= R_1^2, \\ \pi_2 &= R_2^2, \end{aligned} \quad (70)$$

so that the constraints can be rewritten

$$\begin{aligned} \mathcal{H}_t^1 &= -i\mu^2 \pi_1 - 4 \sin^2(\psi_1) = 0, \\ \mathcal{H}_t^2 &= -i\mu^2 \pi_2 - 4 \sin^2(\psi_2) = 0. \end{aligned} \quad (71)$$

B. Quantization

Quantizing by Dirac's procedure, we promote the constraints to operator equations to define the physical state space,

³But we have indeed made some choices in the formulation of the constraints, which make no difference to the classical system, but which do affect details the quantum system beyond the WKB approximation. See Sect. IV for further elaboration.

$$\begin{aligned}\mathcal{H}_t^1 |\Psi\rangle &= 0, \\ \mathcal{H}_t^2 |\Psi\rangle &= 0.\end{aligned}\tag{72}$$

We thus take momenta as differential operators

$$\begin{aligned}\pi_1 &\rightarrow -i\frac{\partial}{\partial i\psi_1} = -\frac{\partial}{\partial\psi_1}, \\ \pi_2 &\rightarrow -i\frac{\partial}{\partial i\psi_2} = -\frac{\partial}{\partial\psi_2},\end{aligned}\tag{73}$$

acting on the wave function $\Psi(\psi_1, \psi_2)$, and we arrive at the two wave equations

$$\begin{aligned}\left[\mu^2\frac{\partial}{\partial\psi_1} + 4\sin^2(\psi_1)\right]\Psi(\psi_1, \psi_2) &= 0, \\ \left[\mu^2\frac{\partial}{\partial\psi_2} + 4\sin^2(\psi_2)\right]\Psi(\psi_1, \psi_2) &= 0,\end{aligned}\tag{74}$$

Since these equations are uncoupled, the wave function separates as the product of 1-wall wave functions, *i.e.*,

$$\Psi(\psi_1, \psi_2) = \Psi_1(\psi_1)\Psi_2(\psi_2).\tag{75}$$

It may seem surprising that the two walls are uncoupled from each other; however this is a necessary consequence of Birkhoff's theorem and conservation of M . We point out two facts: First, the two walls are *topologically* coupled by residing in the same space; second, there will be a local effective coupling between the two walls (*i.e.*, a δ -function coupling) which will mediate their annihilation into pure field energy, that we have missed in the thin-wall approximation. Then our wave equations become

$$\left[\mu^2\frac{\partial}{\partial\psi_j} + 4\sin^2(\psi_j)\right]\Psi(\psi_j) = 0.\tag{76}$$

C. Boundary Conditions

We now set appropriate boundary conditions for the wave functions whose dynamics are given exactly by Eq. (76). These equations take the slightly unusual form of first-order equations, not the usual second-order Schrödinger equation. First-order equations can, however, be routinely handled by Dirac quantization [9].⁴ It is clear that, in the tunnelling regime, solutions will die exponentially as ψ_j increases. The usual freedom to

⁴A slightly different, second-order form of the action will be presented in Sect. IV. There is also still another way to proceed that also leads to second-order Schrödinger equations. We can put \pm signs on the radicals in Eq.(67), to impose two separate constraints at each wall instead of one. Classically, a solution would be such that one of these constraints vanishes; the wave function,

choose either exponentially dying or exponentially growing solutions appears in a slightly different way here: We can choose a ψ_j in the initial state corresponding to the desired value of R_j , and then evolve either to increasing ψ_j or to decreasing ψ_j , respectively. Since our problem involves tunnelling, we wish to evolve both ψ_j to increasing values.

We now restore the factors of G , where

$$\mu^2 G^3 \sim \left(\frac{m_{\text{gut}}}{m_{\text{pl}}} \right)^6. \quad (78)$$

The particular problem we wish to solve involves a pre-existing wall at the turning point $R_1 = 2/\mu G$, and a second wall just nucleating at zero size, $R_2 = 0$. Classically from Eq. (69),

$$R_j = \frac{2}{\mu G} \sin \psi_j, \quad (79)$$

so translating the R_j to canonical wall coordinates,

$$\begin{aligned} \psi_1 &= \pi/2, \\ \psi_2 &= 0, \end{aligned} \quad (80)$$

which give the initial conditions on the wave function.

$$\Psi(\pi/2, 0) = c_{\text{nuc}}, \quad (81)$$

where c_{nuc} is the amplitude to nucleate a zero-size wall, which we take to be ~ 1 .

The tunnelling process then proceeds with both ψ_j increasing, until the walls meet and annihilate. This necessitates for the final state

$$R_1 = R_2, \quad (82)$$

or (taking into account the multiple-valuedness of \sin)

$$\psi_1 = \begin{cases} \psi_2 \\ \pi - \psi_2 \end{cases} \quad (83)$$

The allowed domain for the canonical coordinates is

$$\begin{aligned} \pi/2 &\leq \psi_1 \leq \pi, \\ 0 &\leq \psi_2 \leq \pi/2 \end{aligned} \quad (84)$$

representing a linear combination of such allowed states, will only be annihilated by the product of the constraints. Hence for the purpose of quantization, we write

$$\mathcal{H}_t^j = \mathcal{H}_t^{i(+)} \mathcal{H}_t^{i(-)} = (\mu R_j)^4 - 4(\mu R_j)^2 + 4 \sin^2(2\psi_j). \quad (77)$$

This eventually leads to equivalent results (up to factor ordering ambiguities) after careful imposition of boundary conditions.

so the final condition must be

$$\psi_1 = \pi - \psi_2. \quad (85)$$

The final state is not a turning point of the classical action, Eq. (69), because the annihilation process is missed in this action, as mentioned above. Presumably a small-scale turning point would appear if we worked beyond the thin-wall approximation, but we will not pursue this point. Figure 2 displays the configuration space of the 2-wall system in the tunnelling regime.

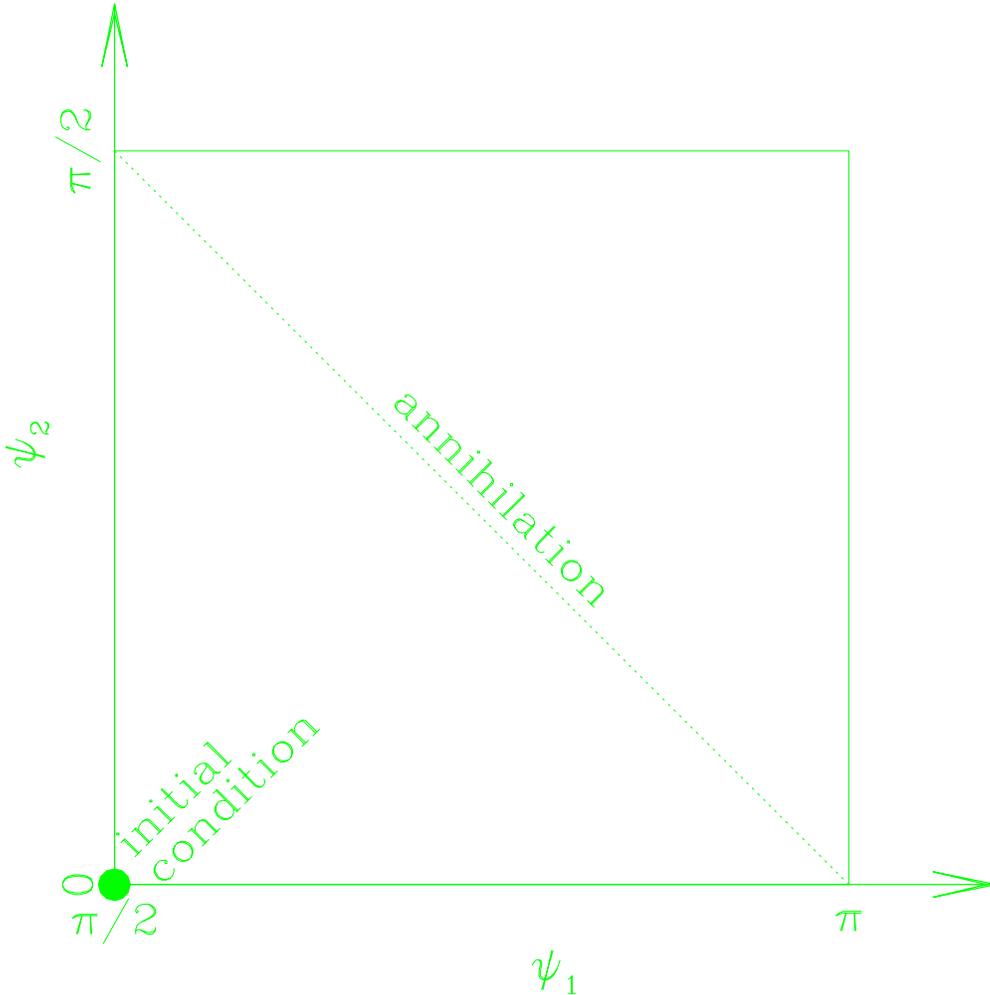


FIG. 2. The configuration space of the two wall system in the tunnelling regime. The coordinates are ψ_1 and ψ_2 . Initial conditions are shown as \bullet at the lower left. The diagonal dotted line represents the locus where the walls collide and annihilate into pure field energy.

D. Solution of the Wave Equations

The wave equations (75,76) comprise two independent first-order equations in the (ψ_1, ψ_2) plane, which are well posed and have a unique solution under the the initial conditions (81)

on the domain (84). This solution, for the wave function of the 2-wall system, is then

$$\Psi(\psi_1, \psi_2) = c_{\text{nuc}} \exp\left(-\frac{1}{\mu^2 G^3} \left[2\psi_1 + 2\psi_2 - \pi - \sin 2\psi_1 - \sin 2\psi_2\right]\right) \quad (86)$$

Evaluating this wave function under the final condition, Eq. (85) gives

$$\Psi_{\text{final}}(\psi_1, \pi - \psi_1) = c_{\text{nuc}} \exp\left(-\pi/\mu^2 G^3\right) \quad (87)$$

for the final state wave function. The corresponding probability of tunnelling to the final state is

$$P_{\text{final}} = |c_{\text{nuc}}|^2 \exp\left(-2\pi/\mu^2 G^3\right) \quad (88)$$

The most remarkable feature of the result of this paper, Eq. (88), is its *independence* of the final value of ψ_1 or ψ_2 . This means that the two walls may collide and annihilate at any value of the final radius in the kinematically allowed range,

$$0 \leq R_{\text{final}} \leq 2/\mu G, \quad (89)$$

with equal probability. At first this may seem surprising, but we argue that it is as expected.

Consider the following toy problem. A particle and an antiparticle move in a potential that is identical for both particles. (For instance, a proton and an antiproton move in a gravitational potential.) There is a potential barrier present, and in the initial state, the two particles are on opposite sides of this barrier. They may tunnel toward each other through the barrier, and annihilate if they meet. The question now is, what is the most probable location for the annihilation? We encourage the reader to stop reading at this point, guess the answer, and then work it out.

The answer is that annihilation is equally probable at any location within the barrier, and the annihilation probability is just given by the total barrier factor for single-particle penetration. We argue that annihilation of the 2-wall system is no different, justifying our result. However, this system is not easy to interpret in the canonical variables $(i\psi, R^2)$. Therefore we will also give a slightly different quantization for the 1-wall system, with application to the 2-wall system, after defining some new canonical variables.

IV. A FURTHER METHOD OF QUANTIZATION

A better pair of variables (Q, P) for the 1-wall system can be obtained by defining

$$\chi = i(\pi/2 - \psi) \quad (90)$$

and then carrying out the following canonical transformation:

$$\begin{aligned} Q &= \sqrt{2}R \cosh \chi, \\ P &= \sqrt{2}R \sinh \chi, \end{aligned} \quad (91)$$

which entail

$$\begin{aligned}
iR^2\dot{\psi} &= P\dot{Q} + (\text{total time derivative}), \\
\mu^2 R^4 - 4R^2 \sin^2 \psi &= \mu^2 R^4 - 4R^2 \cosh^2 \chi, \\
&= \frac{\mu^2}{4}(Q^2 - P^2)^2 - 2Q^2, \\
&= \left(\frac{1}{2}(Q^2 - P^2) + \frac{\sqrt{2}}{\mu}Q \right), \left(\frac{1}{2}(Q^2 - P^2) - \frac{\sqrt{2}}{\mu}Q \right). \tag{92}
\end{aligned}$$

(We are again setting $G = 1$.) The 1-wall action then can be rewritten as

$$\begin{aligned}
S &= \int dt \left\{ iR^2\dot{\psi} - \tilde{N}^t \left[\mu^2 R^2 - 4\sqrt{1 - 2M/R_1} \sin^2 \psi - \left(1 - \sqrt{1 - 2M/R_1} \right)^2 \right] \right\} \\
&= \int dt \left\{ P\dot{Q} - \bar{N}^t \left(\frac{1}{2}(Q^2 - P^2) - \frac{\sqrt{2}}{\mu}Q \right) \right\} \equiv \int dt \{ P\dot{Q} - \bar{N}^t \bar{\mathcal{H}}_t \}, \tag{93}
\end{aligned}$$

where the constraint is redefined by

$$\begin{aligned}
\tilde{\mathcal{H}}_t &= \bar{\mathcal{H}}_t R^{-2} \left(\frac{1}{2}(Q^2 - P^2) + \frac{\sqrt{2}}{\mu}Q \right), \\
\bar{N}^t &= \tilde{N}^t R^{-2} \left(\frac{1}{2}(Q^2 - P^2) + \frac{\sqrt{2}}{\mu}Q \right). \tag{94}
\end{aligned}$$

Once more, the extra factors introduced into the constraint by Eq. (94) never vanish in the classical regime, and make no difference to the classical equations of motion; and they never vanish in the quantum tunnelling regime and so at most affect the prefactor in the tunnelling calculation. The phase space (Q, P) of the 1-wall action in the form (93) can be described as follows. The classically allowed regime is

$$Q \text{ real, } 0 \leq Q < \infty; \quad P \text{ real, } 0 \leq |P| < Q; \tag{95}$$

while the classically forbidden, or quantum tunnelling, regime is

$$Q \text{ real, } -\infty \leq Q < \infty; \quad P \text{ imaginary, } -i\infty \leq P < i\infty. \tag{96}$$

The two regimes meet at $P = 0$.

The action (93) is now entirely straightforward to quantize as a 1-dimensional particle system. We take the 1-wall wave function as $\Psi(Q)$ and use

$$P \rightarrow -i \frac{\partial}{\partial Q} \tag{97}$$

to write the constraint as a Schrödinger equation $\mathcal{H}_t \Psi = 0$ or

$$\left(-\frac{\partial^2}{\partial Q^2} + V(Q) \right) \Psi(Q) = 0, \tag{98}$$

where the potential is an “upside-down harmonic oscillator”.

$$\begin{aligned}
V(Q) &= -Q^2 + \frac{2\sqrt{2}}{\mu}Q, \\
&= -\left(Q - \frac{\sqrt{2}}{\mu}\right)^2 + \frac{2}{\mu^2}.
\end{aligned}
\tag{99}$$

Note that the constraint says that Ψ must be the zero eigenfunction of \mathcal{H}_t ; the rest of the spectrum of \mathcal{H}_t is not present. Equation (98) is hypergeometric and its solutions are parabolic cylinder functions, but we will not pursue the details. Figure 3 shows the potential $V(Q)$ and the resultant dynamics.

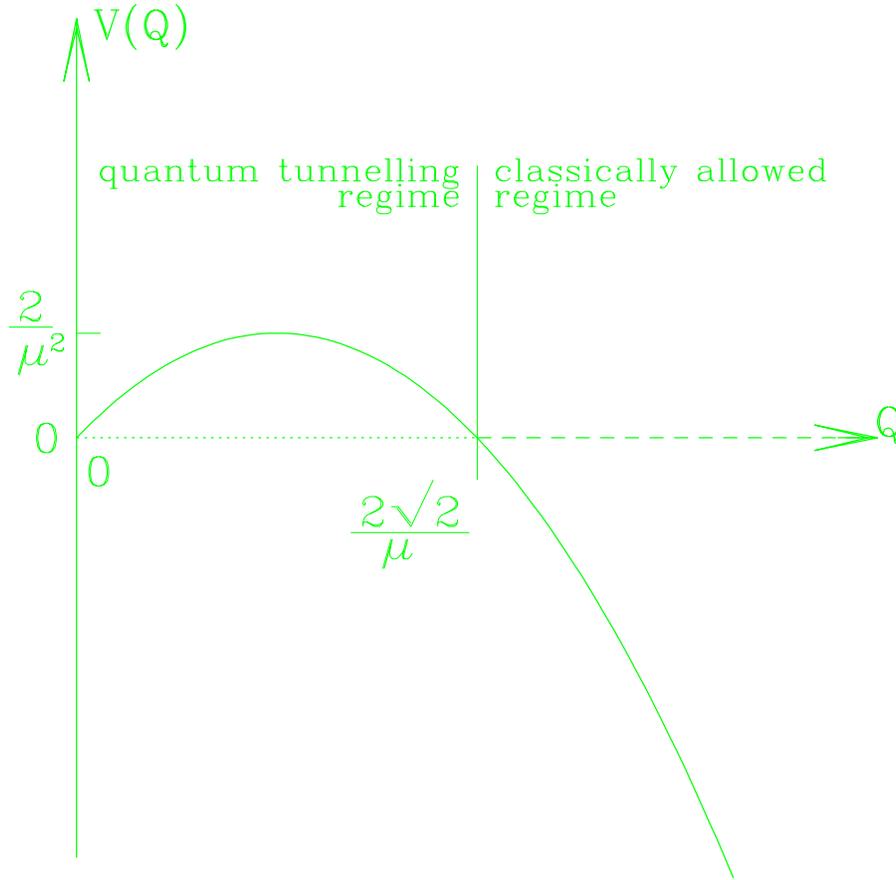


FIG. 3. Dynamics of the quantized VIS solution. The domain wall can be viewed as a particle moving in one dimension Q , under the influence of a potential $V(Q)$ (curve). The energy is constrained to be 0. The turning point is at $Q = 2\sqrt{2}/\mu$; to its right is the classically allowed regime, and to its left is the classically forbidden, or quantum tunnelling, regime. The dashed horizontal line represents the classical motion of the VIS wall, and the dotted line represents quantum tunnelling.

Tunnelling from $Q = 0$ to $Q = 2\sqrt{2}/\mu$ is the “creation of the VIS universe from nothing”, while tunnelling in the opposite direction is “annihilation of the VIS universe into nothing” (*cf.* [23]).

To study the quantum decay of the VIS universe into a universe containing pure field energy, our subject in this paper, two copies of the 1-wall system must be coupled to make the 2-wall system with configuration space (Q_1, Q_2) . Boundary conditions are as follows: For wall 1, impose along the Q_1 -axis purely right-going boundary conditions on the left. For wall 2, impose along the Q_2 -axis purely left-going boundary conditions on the right. Annihilation can occur at any $Q_1 = Q_2$ between the endpoints 0 and $2\sqrt{2}/\mu$. The WKB approximation gives the same exponential barrier factor $P \sim \exp(-\pi/\mu^2 G^3)$ as in Eq. (88) above, and the same main result appears, that the decay probability is independent of wall radius at annihilation (or final Q). We will not pursue the details beyond the WKB regime; presumably the prefactor will differ.

V. CONCLUSION

The main results of this paper are:

1. The 1-wall VIS universe does undergo quantum decay into a universe containing pure field energy, with some small probability.
2. The decay process can be treated as the nucleation of a second domain wall at zero size, followed by quantum tunnelling of the two walls toward each other, and annihilation when they meet.
3. The 2-wall system can be treated in a Hamiltonian approach, using a simple action in the 2-wall phase space, and Dirac quantization.
4. The decay probability for the VIS universe is independent of the radius of the final universe (up to prefactors), and is given by Equation (88).

Conclusions 1 and 2 agree with paper I [1], which employs an instanton approach to replace 3. However, conclusion 4 differs strongly from paper I, which predicted a unique value

$$R_{\text{final}} = \sqrt{2}/\mu G \tag{100}$$

for the annihilation radius. Furthermore, the probability differs: The above value, Eq. (88) differs from the value of paper I, Eq. (I.88), which is

$$P_{\text{paper I}} \sim \exp\left(-\frac{2}{\mu^2 G^3}\right). \tag{101}$$

We interpret this disagreement as an incorrect result of the instanton approach to this problem (at least as done in paper I).

In fact, the two results can be reconciled, if we imitate the Hamiltonian calculation and flip some signs in the instanton calculation, in a way that seems *ad hoc* in the instanton context. In particular, if the four segments in the $n = 2$ instanton are weighted $(+1, -1, +1, -1)$

in calculating the action (see paper I), rather than using the $(+1, +1, +1, +1)$ that was previously motivated both by the standard methodology of Euclidean quantum gravity, and by the 2-sheeted manifold rule of Farhi, Guth and Guven [24]. These sign flips are now motivated by a careful consideration of exponentially growing versus exponentially dying wave functions. After the signs are flipped, the instanton decay probability agrees with Eq.(88). We conclude that the instanton method as utilized in paper I makes incorrect choices for these wave functions. We would like to be able to propose a “modified rule” for sign weights in the instanton calculation that would repair this defect, but have been unable to find a convincing formulation.

We leave the correct instanton treatment of the quantum decay of domain walls as an open problem.

Our result (88) also disagrees with the answer one obtains by assuming that the tunnelling probability is of the form

$$P \sim \exp(I_f - I_i), \tag{102}$$

where I_f and I_i are the Euclidean actions of the instantons which mediate the creation from nothing of the final state and the initial state, respectively, as is often done. But, our answer does happen to be the same as the probability $\exp(-I_j)$ for creation of the initial state alone from nothing, as calculated by using the Euclidean VIS solution as the instanton [23]. It is not clear why the various methods do not agree.

The barrier factor we have calculated is a function of the dimensionless parameter $\mu^{-2}G^{-3} \sim (m_{\text{gut}}/m_{\text{pl}})^6$. For a typical theory, $m_{\text{gut}} \sim 10^{14} - 10^{18}$ GeV; hence $\mu^{-2}G^{-3}$ is expected to be extremely small in most phenomenologically viable models of microphysics. However, improbable events can be important in early-universe cosmology, if they lead to a universe resembling our own.

The new universe created by the decay does not yet resemble our own, however. To do so it must first expand greatly, and then it must homogenize itself. Whether it does so will be the subject of a future paper in this series.

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APPENDIX A: QUANTIZATION OF GAUGE THEORIES

Dirac first worked out the theory of quantizing constrained systems in general [9], and general relativity in particular [10], and his pioneering work continues to serve as the foundation of current efforts to canonically quantize gravity. What follows will be a very brief review of the main elements of such a quantization scheme, sufficient for the purposes of the current study. Many more extensive studies of the subject can be found in the literature; see, for example, [11,25,26].

Consider a mechanical system with n degrees of freedom, which is described by the Lagrangian $L(q_i, \dot{q}_i)$, and where the $q_i(t)$, $i = 1, \dots, n$ are generalized coordinates. The canonical momenta are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(q_i, \dot{q}_i). \quad (\text{A1})$$

To put this action into Hamiltonian form, one seeks to eliminate the velocities \dot{q}_i in favor of the momenta p_i through the use of Eqs. (A1). However, in the event that the Hessian matrix of L with respect to the velocities has zero determinant, *i.e.*,

$$\text{Det} \mathcal{H}_{ij} = \text{Det} \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = 0, \quad (\text{A2})$$

then not all of the \dot{q}_i can be eliminated in this manner. (This will occur, for example, if the action is linear in one or more of the velocities, and quadratic in the rest.) In fact, one can eliminate exactly R of the \dot{q}_i , where $R < n$ is the rank of \mathcal{H}_{ij} . After doing so, one is left with a set of $(n - R)$ constraint equations of the form

$$C_\alpha \equiv p_\alpha - f_\alpha(q_i, p_i) = 0, \quad \alpha = 1, \dots, n - R, \quad (\text{A3})$$

which are known as the *primary* constraints of the theory.

The canonical Hamiltonian,

$$H_c \equiv \sum_{i=1}^R p_i \dot{q}_i - L \quad (\text{A4})$$

is not unique on the full phase space (q_i, p_i) , and so one defines the new Hamiltonian

$$H = H_c + \lambda^\alpha C_\alpha, \quad (\text{A5})$$

where the λ^α are arbitrary functions, or Lagrange multipliers.

Introduce the notation $\{u, v\}$ as the Poisson bracket of the functions $u(p, q)$ and $v(p, q)$, and let $\{u, v\}'$ denote a Poisson bracket to which the constraints have been applied after the calculation of the bracket. Then one divides the primary constraints into two classes, according to the algebra of their Poisson brackets. Those constraints whose Poisson bracket algebra closes, *i.e.*, for which

$$\{C_\alpha, C_\beta\} = f_{\alpha\beta}^\gamma C_\gamma, \quad (\text{A6})$$

or

$$\{C_\alpha, C_\beta\}' = 0, \quad (\text{A7})$$

are known as *first-class* constraints, and all other constraints are known as *second-class*. We will denote a second-class constraint with a Latin index, *e.g.*, C_a . An important fact is that for each first-class constraint there is a corresponding gauge symmetry of the theory.

Since the constraints should hold at all times, we require that

$$\dot{C}_\alpha = \{C_\alpha, H\}' = 0, \quad (\text{A8})$$

which may in some cases lead to inconsistent equations, or which may lead to new relations between the phase space variables. In the latter case, these new relations are known as *secondary* constraints, which are then classified as first-class or second-class as described above. The consistency conditions (A8) must then be checked again, and the process repeated until all constraints have been found.

In order to quantize the theory, we would like to replace the Poisson bracket in the classical relations by $-i/\hbar$ times the commutator of the corresponding quantum operators, and then impose the constraints as conditions on the state vectors. However, note that

$$\begin{aligned} C_\alpha|\psi\rangle = 0, \quad C_\beta|\psi\rangle = 0 \\ \Rightarrow [C_\alpha, C_\beta]|\psi\rangle = 0, \end{aligned} \quad (\text{A9})$$

which corresponds to the classical relation

$$\{C_\alpha, C_\beta\}' = 0. \quad (\text{A10})$$

Hence all of the constraints should be first-class in order for the quantization to go through in a straightforward way.

The prescription for eliminating the second-class constraints is as follows. The Dirac bracket is defined by

$$\{A, B\}^* = \{A, B\} - \{A, C_a\}\Gamma^{ab}\{C_b, B\} \quad (\text{A11})$$

where the algebra of the second-class constraints is

$$\Gamma^{ab}\{C_b, C_c\} = \delta_c^a. \quad (\text{A12})$$

This is a projection of the Poisson bracket onto the second-class constraint surface; therefore if the Poisson brackets are replaced with Dirac brackets in the classical analysis, we can consistently take $C_a = 0$ to hold as operator equations in the quantum theory. Finally the quantization can proceed, now with the Dirac bracket taking the role of the Poisson bracket in the classical theory:

$$\begin{aligned} \{u, v\}^* &\rightarrow -\frac{i}{\hbar}[u, v], \\ p_i &\rightarrow \frac{\hbar}{i} \frac{\partial}{\partial q_i}, \\ C_\alpha(p_i, q_i)|\psi\rangle &= 0. \end{aligned} \quad (\text{A13})$$

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