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Quantum Decay of Domain Walls In Cosmology I: Instanton Approach

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Abstract

This paper studies the decay of a large, closed domain wall in a closed universe. Such walls can form in the presence of a broken, discrete symmetry. We introduce a novel process of quantum decay for such a wall, in which the vacuum fluctuates from one discrete state to another throughout one half of the universe, so that the wall decays into pure field energy. Equivalently, the fluctuation can be thought of as the nucleation of a second domain wall of zero size, followed by its growth by quantum tunnelling and its collision with the first wall, annihilating both. The barrier factor for this quantum tunneling is calculated by guessing and verifying a Euclidean instanton for the two-wall system. We also discuss the classical origin and evolution of closed, topologically spherical domain walls in the early universe, through a "budding-off" process involving closed domain walls larger than the Hubble radius. This paper is the first of a series on this subject.

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I. INTRODUCTION

Following the classic paper of Zel'dovich, Kobsarev & Okun [1], it has been generally believed in early-universe cosmology that domain walls are forbidden, because otherwise they would gravitationally dominate the present-day universe, contrary to observation. Consequently, fundamental theories of physics are either forbidden to exhibit spontaneously broken discrete symmetries, or are constrained to eliminate the associated domain walls somehow, for instance through low energy destabilization of all but one of the discrete vacuua, or through decay of the domain walls themselves through nucleation of string loops into walls. Only then might an early universe dominated by domain walls evolve into a present-day universe consistent with current cosmological observations.

The purpose of this paper is to discuss a novel process by which domain walls can decay, namely, the quantum decay of domain walls by global fluctuation and quantum tunneling. We introduce and study this process for closed universes containing a single domain wall, but the decay process itself is no doubt more general. Imagine a closed universe separated into two different discrete vacuum states by a single closed domain wall. If one of these vacuum states spontaneously fluctuates into the other, the domain wall will cease to exist and energy will be liberated. Equivalently, the fluctuation can be thought of as the nucleation of a second domain wall of zero size, followed by its growth by quantum tunnelling and its collision with the first wall, annihilating both. Clearly this decay will be slow since it involves a fluctuation over an entire half of the universe, and moreover it involves curved spacetime, and therefore requires semiclassical quantum gravity for its description.

Such a closed universe can be created, for instance, by the the gravitational collapse of closed (*i.e.,* bubble-like) domain walls. During a cosmological phase transition that produces domain walls by the Kibble mechanism, both open and closed domain walls will be produced. Closed domain walls collapse due to their surface tension. A collapsing closed domain wall may thermalize completely, leaving no remnant, or it may produce a black hole by gravitational collapse. We especially consider in this paper the behavior of closed domain walls that happen to be born with size R_0 larger than the then-Hubble radius R_H. These will originate naturally with some probability in the Kibble mechanism. As one might expect, such a closed domain wall will typically produce a black hole. However, the spacetime singularity inside the black hole may not destroy the domain wall; rather, the domain wall may expand indefinitely to create a new, inflating universe within the black hole, in a "budding-off" process. This process was briefly considered by Blau et al. [2], but has not attracted interest since then. Studying a simple but appropriate model, we conclude that new-universe creation is obligatory in the model, as long as $R_0 \gtrsim R_H$ at the time of the phase transition, and we argue that this conclusion is apt to be robust.

The newly created inflating universe is dominated by a single closed domain wall, and therefore does not resemble our present universe. Since the domain wall is of finite size, however, it is subject, as mentioned above, to decay by quantum tunneling, unlike an infinite domain wall, which cannot decay because of an infinite barrier against quantum tunneling. After a long time, inflation will therefore end by quantum tunneling of the whole domain wall, followed by thermalization of its energy. The resulting universe will therefore exhibit a hot big bang, and may conceivably resemble our universe.

A comprehensive discussion of domain walls in cosmology is given by Vilenkin and Shellard [3]. There has been a great deal of work on formation and dynamics of closed domain walls between domains of true and false vacuua — the so-called "bubble walls" — including the possibility of budding off of new universes within a newly formed black hole. The present work differs from that work in that we never appeal to a false vacuum; all domain walls in this paper occur between regions of true vacuum, as a result of a spontaneously broken, exact, discrete symmetry. There has been some very interesting work on "topological inflation" within defects including domain walls, cosmic strings or monopoles [4,5]. The present work differs from that in that we consider quantum decay of an entire domain wall, while that work depends on quantum fluctuations within a domain wall, monopole, or cosmic string. Recently, Caldwell, Chamblin, and Gibbons [6] studied the pair production of black holes in the spacetime of a domain wall, using instanton methods. That process is related to, but distinct, from the one we study here.

In Section II, we discuss some basic properties of domain walls and derive the Euclidean action for walls in the thin-wall approximation. Section III reviews the solutions describing the gravitational fields of open (Vilenkin) and closed (Ipser-Sikivie) domain walls, and discusses their behavior. Section IV presents a complete discussion of the gravitational fields and motion of domain walls in spherical symmetry, assuming true vacuum everywhere except for the domain wall itself. Section V sets up a spherically symmetric cosmological model in which a single closed domain wall forms, and demonstrates the conditions under which this domain wall can create a new inflating universe. Inflation in this new universe is driven by the domain wall that creates it. Section VI contains the heart of our development and discusses the quantum decay of the domain wall in a closed, domain wall-dominated inflating universe, and estimates the probability per unit time — small but nonvanishing — for the decay of the domain wall by tunneling processes in semiclassical quantum gravity.

We emphasize that all the processes considered in this paper — budding-off, inflation, creation of cosmological fluctuations, and quantum wall decay, are driven by domain walls. False vacuum energy plays no role. This paper is the first of a series on this subject; the second paper will study the same quantum decay process by a different method, namely a Hamiltonian approach.

II. THE ACTION FOR DOMAIN WALLS

Domain walls appear in matter field theories where discrete symmetries are spontaneously broken. The action for matter plus gravity is of the form

$$S = \int_M d^4x \sqrt{-g} \left[L_{\rm mat} + \frac{R}{16\pi G} \right] + \int_{\partial M} d^3x \frac{\sqrt{h} K_h}{8\pi G}. \tag{1}$$

where the 4-volume M of the system may have a 3-boundary ∂M, with 3-metric h and 3-extrinsic curvature of trace K_h.[1] Little of this paper will depend on the details of the

[1]Our conventions are generally those of MTW [7]: Greek spacetime indices run over $\mu, \nu, \ldots = 0, 1, 2, 3$ and the spacetime signature is $(-,+,+,+)$; Roman indices in a hypersurface run over $a, b, \ldots = 1, 2, 3$ or $a, b, \ldots = 0, 1, 2$ for spacelike or timelike hypersurfaces respectively; our extrinsic

matter, but for concreteness, the matter may be chosen as a real scalar field ϕ with matter action

$$L_{\rm mat} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \tag{2}$$

where $U(\phi)$ has a discrete number of degenerate minima at which $U = 0$, *e.g.,* $U(\phi)$ may be a "double well" potential.

A. The Euclidean action

The Euclidean action I is obtained by analytically continuing Eq. (1) to imaginary time and then reversing its sign, *i.e.,*

$$I = -\int_M d^4x \sqrt{g} \left[L_{\rm mat} + \frac{R}{16\pi G} \right] - \int_{\partial M} d^3x \frac{\sqrt{h} K_h}{8\pi G} \tag{3}$$

where the metric is now positive definite. We assume that $L_{\rm mat}$ has a discrete broken symmetry and therefore exhibits domain walls as solitons in the low energy limit, without any other light degrees of freedom. In this paper, we treat domain walls in the "thin-wall" approximation [8], where the thickness of the domain wall is assumed much smaller than all other length scales in the problem, so that the wall may be taken as a two-dimensional sheet of stress-energy in 3-space. The low energy Euclidean action in the thin-wall approximation is

$$I_{tw} = -\int d^4x \frac{\sqrt{g} R}{16\pi G} + \sigma \sum_i \int_{D_i} d^3x \sqrt{h} - \int_{\partial M} d^3x \frac{\sqrt{h} K_h}{8\pi G} \tag{4}$$

where the sum on i runs over some finite number of separate domain walls D_i. The action of each wall is simply proportional to its 3-volume, with the surface energy density σ a constant, fixed by microphysics of $L_{\rm mat}$. In the thin-wall approximation, the spacetime geometry is generally not smooth across each wall; rather there is a 3-dimensional δ-function in the Riemannian curvature at each wall [9]. This renders the action (4) awkward to work with. Therefore, we will eliminate these δ-functions by breaking up M into a union of 4-dimensional voids M_j, each with smooth interior 4-geometry, meeting at domain walls D_i. The Euclidean action becomes

$$I_{tw} = -\sum_j \int_{M_j}' d^4x \frac{\sqrt{g} R}{16\pi G} + \sum_i \int_{D_i} d^3x \sqrt{h} \left[\sigma + \frac{(K_2 - K_1)}{8\pi G} \right] - \int_{\partial M} d^3x \frac{\sqrt{h} K_h}{8\pi G} \tag{5}$$

where the prime on the integral $\int'$ means that any δ-function in R at a wall does *not* contribute to the 4-integral; and where h_{ab} is now the 3-metric on each domain wall D_i (as well as the fixed 3-metric on the system boundary ∂M), and K_1 and K_2 are the traces of the

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curvature K_{ab} is the derivative $\frac{1}{2}\mathcal{L}_{\mathbf{n}} h_{ab}$ of the hypersurface 3-metric h_{ab} with respect to an outward pointing normal vector $\mathbf{n}$. Thus, $R > 0$ and $K > 0$ for a sphere embedded in flat space.

extrinsic curvature of each domain wall on its two sides, with respect to the normal vector that points from 1 to 2. Due to the unsmoothness of the spacetime geometry, $K_1 \neq K_2$. The instrinsic 3-metric h_{ab} of each domain wall is constrained to agree with that inherited from the 4-geometry $g_{\mu\nu}$ from the void on each of the two sides.

The field equations become as follows. Variation of $g_{\mu\nu}$ within each M_j gives the the vacuum Einstein equations

$$G_{\mu\nu} = 0 \quad (\text{within } M_j) \tag{6}$$

within each void. Variation of the domain wall metric h_{ab} on each domain wall D_i gives the well known Israel jump condition [9] for the full extrinsic curvature K^{ab}

$$K_1^{ab} - K_2^{ab} = 4\pi G\sigma h^{ab} \tag{7}$$

showing that there is a jump $4\pi\sigma$ in the trace of the extrinsic curvature at the domain wall, while there is no jump in the traceless part of the extrinsic curvature.

B. The action of a solution

The solutions of the field equations are the extrema of the action. After the action is varied and a solution found, simplifications occur in the formula for value of the action at the solution. Substituting the equations of motion (6,7) back into the action (5) give the (weak in the Dirac sense) formula

$$I_{tw} = -\sum_i \frac{\sigma}{2} \int_{D_i} d^3x \sqrt{h} \tag{8}$$

showing that the thin-wall action of a domain wall is always negative.

This formula was given by Caldwell, Chamblin, and Gibbons [6] for the particular case of a real scalar field theory (2), as follows. The stress-energy tensor for the matter is obtained in the canonical way and is found to be

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_{\text{mat}})}{\delta g^{\mu\nu}} \\ &= -\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left[\frac{1}{2} g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + U(\phi)\right] \end{aligned} \tag{9}$$

from which one calculates, taking the trace of the Einstein equations,

$$\frac{R}{8\pi G} = g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 4U(\phi), \tag{10}$$

so that the Euclidean action of a solution to general relativity reduces to

$$I = -\int \sqrt{g} d^4x U(\phi). \tag{11}$$

In the thin-wall approximation, $U(\phi) = 0$ except very near a wall, where it behaves like a δ-function. In a small neighborhood of a point on a wall, the field is approximately planar and is only a function of the coordinate x^3 normal to the wall in a Gaussian normal coordinate system. Then the equation of motion for the field has the first integral

$$\frac{1}{2}\left(\frac{d\phi}{dx^3}\right)^2 - U(\phi) = 0, \tag{12}$$

so that Eq. (9) gives

$$T_\mu^\nu = 2U(x^3)\mathrm{diag}(1,1,1,0). \tag{13}$$

Furthermore, in this approximation, the stress-energy also has a δ-function singularity at the wall [10,2], so that

$$T_\mu^\nu = \sigma \mathrm{diag}(1,1,1,0)\delta(x^3), \tag{14}$$

i.e., $2U(x^3) = \sigma\delta(x^3)$, where σ is the surface energy density of the wall, as well as its tension in the tangential directions. Combining this result with Eqs. (3, 11), one again finds Eq.(8) as the action of a domain wall solution to the field equations.

For convenience we will adopt the notation

$$\mu \equiv 4\pi\sigma \tag{15}$$

in what follows.

III. THE VIS SOLUTIONS FOR WALL-DOMINATED UNIVERSES

A dynamical solution to Einstein's equations with a single thin domain wall present was found by Vilenkin in [11] and by Ipser and Sikivie in [10]. We will henceforth refer to this solution as the VIS solution. One can construct this solution from flat spacetime

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2 \tag{16}$$

as follows. The domain wall itself is the hyperboloid

$$X^2 + Y^2 + Z^2 - T^2 = \left(\frac{2}{\mu G}\right)^2, \tag{17}$$

and the "gravitational field" of the domain wall is the interior

$$X^2 + Y^2 + Z^2 - T^2 < \left(\frac{2}{\mu G}\right)^2. \tag{18}$$

The complete solution is constructed from two such copies of flat spacetime, glued together along their hyperboloids. The entire spacetime is therefore spatially closed, with topology S^3 in the spatial sections. This spacetime can be interpreted as a closed cosmology, containing a single spherical domain wall. The universe collapses from infinite size, halts at a minimum radius $R_{min} = \frac{2}{\mu G}$ due to self-repulsion of the wall, and expands back out to infinite size again.

The Vilenkin solution is actually a subset of the above S^3 spacetime in a different coordinate system, and moreover it is conventionally given the somewhat different interpretation of an infinite planar wall. It is built out of the spacetime:

$$ds^2 = \left(\frac{2}{\mu G}\right)^2 dz^2 + z^2[-dt^2 + \exp(\mu G t)(dr^2 + r^2 d\phi^2)]. \tag{19}$$

The complete Vilenkin solution is, similarly, two copies of this spacetime for $z < \frac{2}{\mu G}$, glued together along the domain wall at $z = \frac{2}{\mu G}$. One can relate the two solutions as follows. Rewrite Eq. (16) slightly as

$$ds^2 = -dUdV + dR^2 + R^2 d\phi^2 \tag{20}$$

where $T = (U+V)/2$, $Z = (-U+V)/2$, $X = R\cos\phi$, and $Y = R\sin\phi$. Then the transformation relating the solutions is:

$$r = \left(\frac{2}{\mu G}\right)\frac{R}{V}, \tag{21}$$

$$z = \left(\frac{\mu G}{2}\right)\sqrt{R^2 - UV}, \tag{22}$$

$$t = \left(\frac{2}{\mu G}\right)\ln\frac{V}{\sqrt{R^2 - UV}}. \tag{23}$$

One can verify that the Vilenkin solution given by Eq. (19) covers only half of the full spacetime, all of which is given by the Ipser and Sikivie result. This is due to the fact that the coordinates in (19) cover only half of the (2+1)-dimensional deSitter space in the hypersurfaces of constant z.

IV. MOTION OF SPHERICAL SYMMETRIC DOMAIN WALLS

In general, domains separated by thin walls may not consist of entirely flat space. In this section, we generalize the VIS solution to include nonzero Schwarzschild masses in each part of the spacetime, and then analyze the possible resulting dynamics of the wall.

The junction conditions of the Einstein equations across a thin shell of stress-energy were first worked out by Israel in [9], who showed that the metric across the surface is continuous, whereas the extrinsic curvature has a jump discontinuity there. In the case of spherically symmetric domain walls, the trajectory and 4-velocity of the wall are specified by

$$x^\alpha(\tau) = (T(\tau), R(\tau), \theta, \phi) \tag{24}$$

$$u^\alpha(\tau) = (\dot{T}(\tau), \dot{R}(\tau), 0, 0) \tag{25}$$

and the junction condition can be written in the form

$$K^{(+)}_{\theta\theta} - K^{(-)}_{\theta\theta} = -\mu G R^2. \tag{26}$$

Taking the spacetime on each side of the wall to be Schwarzschild-deSitter, with line element

$$ds^2 = -\left(1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}\right) dT^2 + \left(1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}\right)^{-1} dR^2 + R^2 d\Omega_2^2, \tag{27}$$

one calculates

$$\begin{aligned} K_{\theta\theta} &= R\left(1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}\right) \dot{T} \\ &= \pm R\left(1 - \frac{2M}{R} - \frac{\Lambda R^2}{3} + \dot{R}^2\right)^{\frac{1}{2}} \end{aligned} \tag{28}$$

where the second equality comes from normalizing the 4-velocity. The junction condition can then be rewritten as

$$\begin{aligned} \dot{R}^2 = &\left[\frac{R(\Lambda_+ - \Lambda_-)}{6\mu} + \frac{(M_+ - M_-)}{\mu R^2}\right]^2 + \frac{(\mu G R)^2}{4} \\ &+ \frac{G(M_+ + M_-)}{R} + \frac{G R^2(\Lambda_+ + \Lambda_-)}{6} - 1, \end{aligned} \tag{29}$$

which is a first order ODE for $R(\tau)$, and can be regarded as an energy-type equation with μ, Λ and M all conserved. The second order equation of motion for $R(\tau)$ can be found by differentiation. Restricting to the true vacuum, one finds

$$\dot{R}^2 = \left[\frac{(M_+ - M_-)}{\mu R^2}\right]^2 + \frac{(\mu G R)^2}{4} + \frac{G(M_+ + M_-)}{R} - 1, \tag{30}$$

and further restricting to the case of flat spacetime, one finds

$$\dot{R}^2 = \frac{(\mu G R)^2}{4} - 1, \tag{31}$$

which is the VIS solution. In what follows we will take $\Lambda = 0$, and study the behavior of domain walls in the absence of vacuum energy, *i.e.,* with dynamics given by Eq. (30).

To examine Eq. (30) qualitatively, change to the dimensionless variables

$$z = R\left[\frac{\mu^2 G}{2(M_+ - M_-)}\right]^{1/3} \tag{32}$$

$$\tau' = \frac{\mu G \tau}{2}. \tag{33}$$

upon which the equation of motion of the wall becomes

$$z'^2 + V(z) = E, \tag{34}$$

where

$$V(z) = -\left[\frac{z^6 + 2z^3(M_+ + M_-)/(M_+ - M_-) + 1}{z^4}\right], \tag{35}$$

$$E = -\left[\frac{4}{\mu G^2(M_+ - M_-)}\right]^{2/3}. \tag{36}$$

Although we will assume $M_+ > M_-$ so that $z > 0$ in what follows, this is a matter only of convention since Eq. (30) is symmetric with respect to M_+ and M_-.

Figure 1 shows a typical plot of $V(z)$, and one sees that there exist several qualitatively distinct classes of solutions, depending upon the values of the parameters (M_-, M_+, μ). A wall may be born with zero size, expand to a finite maximum radius and recollapse; it may be born with zero size and expand indefinitely; it may collapse from infinite size to a minimum radius and then reexpand (this is the behavior of the VIS solution); or it may collapse from infinite size to zero size. In each case, the complete spacetime consists of two pieces of the extended Schwarzschild solution, glued together at the wall.

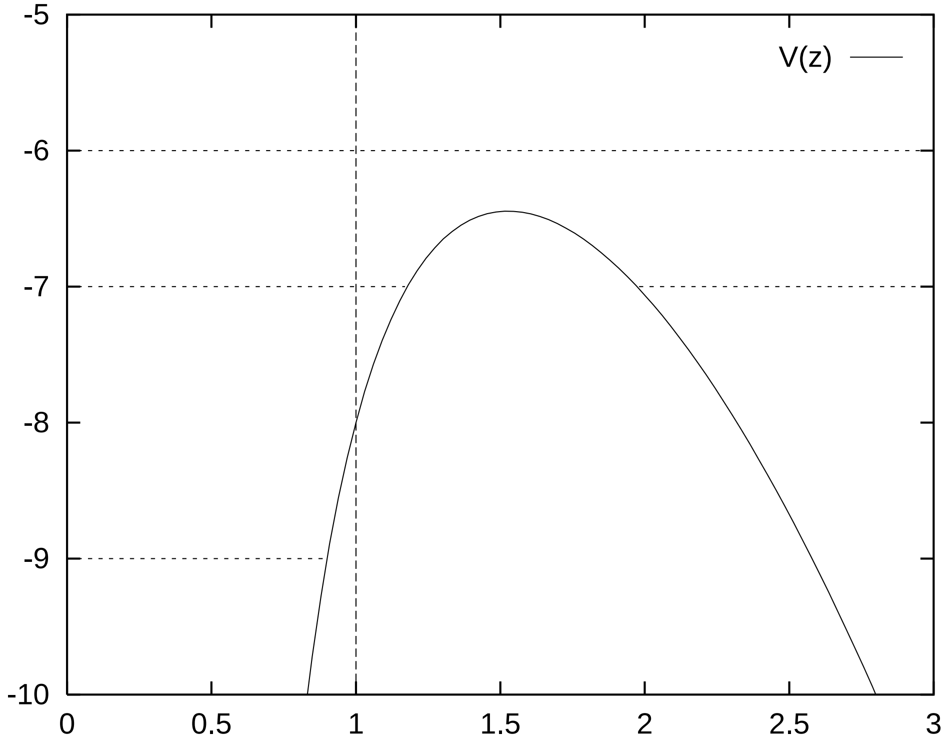

FIG. 1. The "potential energy" function $V(z)$ for domain walls moving in Schwarzschild spacetime. When a trajectory crosses the vertical dashed line at $z = 1$, then the polar angle (angle between the origin and the trajectory) in one half of the complete spacetime changes sign. The horizontal lines represent the possible qualitatively distinct classes of dynamics.

One can construct these spacetimes as follows. Change to coordinates which are well-behaved at the Schwarzschild horizon, and in which light rays travel at 45 degrees. Then there is a correspondence between the sign of the extrinsic curvature in Eq. (28) and the sign of the wall trajectory's "angular velocity", $\frac{d}{dt}\arctan(x^0/x^1)$ in those coordinates. For example, in the Kruskal-Szekeres [12] coordinates (v, u, θ, ϕ), which are related to the standard Schwarzschild coordinates by

$$u^2 - v^2 = \left(\frac{R}{2M} - 1\right) e^{\frac{R}{2M}} \tag{37}$$

$$T = \begin{cases} 4M \mathrm{arctanh}\left(\frac{v}{u}\right) & R > 2M, \\ 4M \mathrm{arctanh}\left(\frac{u}{v}\right) & R < 2M, \end{cases} \tag{38}$$

one finds

$$\begin{aligned} K^{+}_{\theta\theta} &= \pm R\left(1 - \frac{2M_+}{R} + \dot{R}^2\right)^{\frac{1}{2}} \\ &= +8M_+^2 e^{-\frac{R}{2M_+}}(u\dot{v} - v\dot{u}); \qquad (39) \\ K^{-}_{\theta\theta} &= \pm R\left(1 - \frac{2M_-}{R} + \dot{R}^2\right)^{\frac{1}{2}} \end{aligned}$$

$$= -8M_-^2 e^{-\frac{R}{2M_-}}(u\dot{v} - v\dot{u}). \tag{40}$$

In these coordinates, the polar angle in the spacetime diagram is given by $\tan\zeta = v/u$, and its rate of change is given by

$$\dot{\zeta} = \frac{u\dot{v} - v\dot{u}}{u^2 + v^2}. \tag{41}$$

Hence we see that when $K^+_{\theta\theta}$ is positive in a given region of the spacetime, the wall trajectory follows a path such that ζ increases along the trajectory, and vice versa. Similarly one sees that when $K^-_{\theta\theta}$ is positive, ζ *decreases* along the wall trajectory, and vice versa. The sign change is due to the fact that the vector normal to the 4-velocity changes its orientation from one side of the wall to the other.

The last step is to determine the signs of the quantities $K^+_{\theta\theta}$ and $K^-_{\theta\theta}$ as a function of z in Fig. 1. One finds that

$$K^+_{\theta\theta} \begin{cases} > 0, & z > 1 \\ < 0, & z < 1 \end{cases} \tag{42}$$

whereas

$$K^-_{\theta\theta} < 0, \qquad z > 0. \tag{43}$$

Finally one knows everything necessary to construct the complete spacetimes from Fig. 1. In each part of the spacetime, one only has to draw the wall trajectory such that it has the correct starting and ending points, and also so that the angle ζ increases or decreases in each part according to the signs of the extrinsic curvature as stated above.

Figure 2 shows the Penrose (conformal) spacetime diagrams for each of the possible cases. In each case, the complete spacetime consists of the wall trajectory, those points in the right-hand copy of Schwarzschild to its right, and those points in the left-hand copy of Schwarzschild to its left. Of particular interest is case (iv), where a wall is born with zero size and grows to infinite size. An observer in the asymptotically flat region IV of the spacetime would see the wall enter the Schwarzschild horizon; however, the wall subsequently avoids the singularity within the horizon and expands indefinitely, creating a new, inflating universe inside the black hole. Clearly, this model suffers from the requirement that an initial singularity exist from which the domain wall emerges. We therefore proceed to explicitly construct a model which circumvents this necessity, while still producing a new wall-dominated universe.

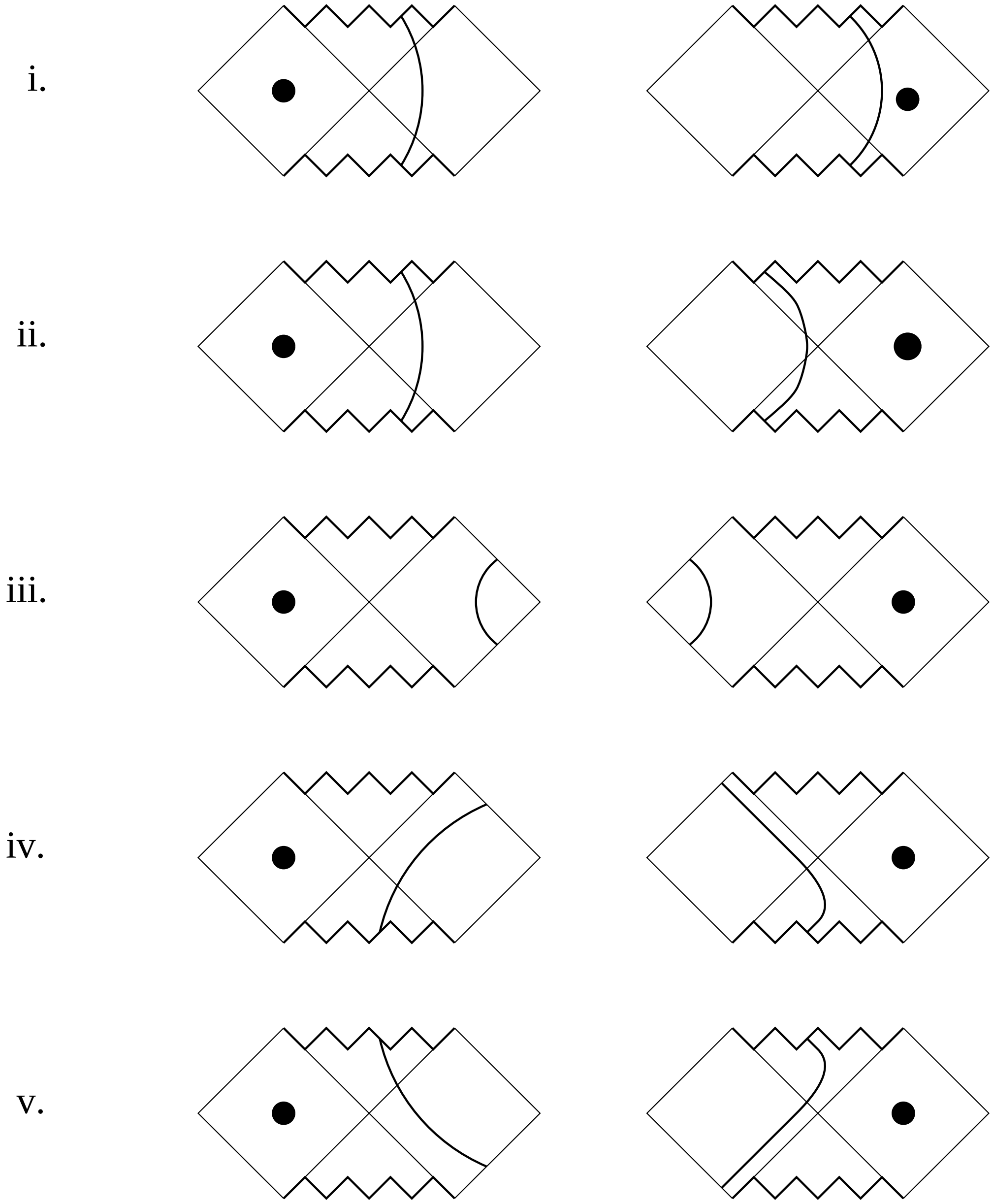

FIG. 2. Spacetime diagrams for each of the possible qualitatively distinct classes of dynamics, for domain walls moving in Schwarzschild spacetime. In each case, the complete spacetime consists of the wall, those points in the right-hand copy of Schwarzschild to its right, and those points in the left-hand copy of Schwarzschild to its left.

V. A MODEL OF NEWLY FORMED BUBBLES OF DOMAIN WALL

We now set up and study a toy model of the behavior of topologically closed and spherical domain walls in cosmology. For convenience we work in units where $G = 1$, in this Section only. Our cosmological model is spherically symmetrical, closed, and bounded. The matter content of the model is pressureless dust, arranged in concentric spherical shells. An important feature of our model is that would be fated to recollapse in a finite time, if it were not for the newly born closed domain wall. This feature is important because we want to avoid giving the newborn domain wall a spurious kick, *i.e.,* a positive total energy. In applications we may think of the model as being only moderately larger than the Planck radius, and with a total lifetime only moderately larger than the Planck time — if it were not for domain wall effects.

(As a limiting case we can study a universe in which each shell starts with marginal ("$k = 0$") binding energy, so that the whole universe just barely expands to infinity as $t \to \infty$.)

In other words, we want our model to be a *purely curvature-free pertubation* of a completely homogeneous pressureless $k = +1$ model, because phase transitions cannot produce correlated pertubations outside the light cone, and therefore, as is well known, cannot produce curvature perturbations.

We concentrate on a single dust shell of finite total proper mass ΔM, idealized as being infinitely thin. [2] Spacetime is vacuum for a small region (see below) around our shell, to avoid complications due to collisions of shells; see below. By Birkhoff's theorem, the spacetime just inside the shell is Schwarzschild of some mass M_1, and that outside is Schwarzschild of some mass M_2. Our shell moves according to the first-order equation of motion

$$\dot{R}^2 = \frac{\Delta M^2}{4R^2} - 1 + \frac{M_2 + M_1}{R} + \frac{(M_2 - M_1)^2}{\Delta M^2} \tag{44}$$

where $R(\tau)$ is the curvature radius of our shell as a function of its proper time τ, and $\dot{=} d/d\tau$. In order that the shell be gravitationally bound, with some relative binding energy ϵ, we take

$$M_2 - M_1 = \epsilon \Delta M \quad (0 < \epsilon \leq 1). \tag{45}$$

Then the equation of motion becomes

$$\dot{R}^2 = \frac{\Delta M^2}{4R^2} + \frac{M_2 + M_1}{R} - 1 + \epsilon^2. \tag{46}$$

If $0 < \epsilon < 1$ our shell would recollapse; in the limit that $\epsilon = 1$ our shell would just make it to infinity — if left undisturbed. However, it is instead going to turn into a domain wall.

To model the birth of a domain wall, our shell is assumed to suddenly become a domain wall as it reaches some radius R_0: a tangental stress equal in magnitude to the surface mass density suddenly appears. However, the mass density (and total mass ΔM) must remain meanwhile remain constant, according to conservation laws. Thereafter, our shell-cum-domain-wall moves according to a different first-order equation of motion,

$$\dot{R}^2 = \frac{1}{4}\mu^2 R^2 - 1 + \frac{(M_2 + M_1)}{R} + \frac{(M_2 - M_1)^2}{\mu^2 R^4} \tag{47}$$

[2] An infinitely thin "dust" shell of finite mass is an acceptable idealization within general relativity; however, subtleties do lurk: A close analysis shows that the shell must possess radial stresses in order to stay infinitely thin — this is for instance clear from from the fact that a dust particle just inside the shell must experience a different acceleration than one just outside.

To match R and $\dot{R}$ at R_0 (these must be continuous) we have

$$\mu = \frac{\Delta M}{R_0^2} \tag{48}$$

Thus the first-order equation of motion of R for the spherical domain wall becomes

$$\dot{R}^2 = \frac{\mu^2 R^2}{4} - 1 + \frac{M_2 + M_1}{R} + \frac{\epsilon^2 R_0^4}{R^4} \tag{49}$$

The main question is simple: How does the solution of this equation $R(\tau)$ behave? Does it go to ∞ as $\tau \to \infty$, or in contrast does it fall back to $R = 0$ at some finite time τ? The former behavior means that the shell has formed a new, inflating, semi-closed universe inside the black hole M_2, while the latter behavior means that the shell has crunched into the singularity inside the black hole M_2. Note that in both cases, the domain wall creates a new black hole of mass M_2 by gravitational collapse, as seen by external observers in the original universe.

Our model therefore has three dimensionful paramaters, μ, M_1, and R_0, and one further dimensionless parameter, ϵ. (By the above, $M_2 = M_1 + \mu\epsilon R^2$.) Physically, μ is fixed by microphysics, R_0 is set through the Kibble mechanism by random variations during the phase transition and is likely to be roughly the Hubble radius then, and M_1 is the mass of matter that happens to lie inside the bubble of domain wall when it forms. The relative binding energy ϵ is set by the overall dynamics of the initial universe, with some further adjustment by the Kibble mechanism.

Since three of the parameters are dimensionful, it is convenient to render them dimensionless by forming ratios with the quantity $M_2 + M_1$, reducing them to two dimensionless parameters, m and r_0, with $0 \leq m \leq 1$ and $0 < r_0 < \infty$:

$$m = \frac{M_2 - M_1}{M_2 + M_1} = \frac{\epsilon\mu R_0^2}{\mu\epsilon R_0^2 + 2M_1}, \tag{50}$$

$$r_0 = \frac{R_0}{M_2 + M_1} = \frac{R_0}{\mu\epsilon R_0^2 + 2M_1}; \tag{51}$$

and a dimensionless radius $r(t)$ as a function of a dimensionless proper time t,

$$r = \frac{R}{M_2 + M_1} = \frac{R}{\mu\epsilon R_0^2 + 2M_1}, \tag{52}$$

$$t = \frac{\tau}{M_2 + M_1} = \frac{\tau}{\mu\epsilon R_0^2 + 2M_1}. \tag{53}$$

In terms of the dimensionless quantities, the equation of motion is

$$r'^2 + V(r) = 0 \tag{54}$$

where the effective potential is

$$V(r) \equiv -\frac{m^2}{4\epsilon^2 r_0^4} r^2 + 1 - \frac{1}{r} - \frac{\epsilon^2 r_0^4}{r^4}. \tag{55}$$

The effective potential $V(r)$ diverges to $-\infty$ as either r$\to 0$ or $r \to \infty$. Therefore $V(r)$ has a maximum at some point r_x which will be studied below; it will turn out that $V(r)$ has only one local maximum for $0 < r < \infty$. If the system point gets to sufficiently large r, a new, semi-closed, inflating universe will be born. The system point starts at $r = r_0$, at the moment when the domain wall is born. The key question is whether it can get past the maximum at r_x. It will get to large r if either of two conditions is satisfied: 1) $r_0 \geq r_x$; or 2) $V(r_x) < 0$.

Let us find the maximum of $V(r)$. The gradient of V is

$$dV/dr = -\frac{m^2}{2\epsilon^2 r_0^4} r + 1/r^2 + \frac{4\epsilon^2 r_0^4}{r^5} \tag{56}$$

and the only positive root of the equation $dV/dr = 0$ is found to be

$$r_x = \frac{r_0^{4/3}\epsilon^{2/3}}{m^{2/3}} \left(1 + \sqrt{1+8m^2}\right)^{1/3}. \tag{57}$$

The condition 1) above that $r_0 \geq r_x$ is then

$$r_0 \leq \frac{m^2}{\epsilon^2(1 + \sqrt{1+8m^2})} \tag{58}$$

The maximum value of the potential is

$$V(r_x) = 1 - \frac{3m^{2/3}}{2\epsilon^{2/3} r_0^{4/3}} \frac{1 + 2m^2 + \sqrt{1+8m^2}}{\left(1 + \sqrt{1+8m^2}\right)^{4/3}}, \tag{59}$$

and condition 2) that this maximum be negative is

$$r_0 < \left(\frac{3}{2}\right)^{3/4} \frac{m^{1/2}}{\epsilon^{1/2}} \frac{\left(1 + 2m^2 + \sqrt{1+8m^2}\right)^{3/4}}{1 + \sqrt{1+8m^2}} \tag{60}$$

Only one of Eqs. (58, 60) need be satisfied for the system to get to large r. Comparing these equations one sees that Eq. (60) dominates (*i.e.,* is less restrictive than) Eq. (58) as long as

$$\epsilon > \left(\frac{2}{3}\right)^{1/2} \frac{m}{(1 + 2m^2 + \sqrt{1+8m^2})^{1/2}} \tag{61}$$

while on the other hand Eq. (58) dominates if this inequality is reversed. In particular, Eq. (61) holds for all permissible m if $\epsilon > 1/3$. So let us concentrate on the case $\epsilon > 1/3$ (shell not greatly bound) and study Eq. (60).

Returning to dimensional variables, the condition Eq. (60) becomes

$$\epsilon\mu^2 R_0^2 + 2\mu M_1 > \left(\frac{2}{3}\right)^{3/2} \frac{\left(\sqrt{1+8m^2}\right)^2}{\left(1 + 2m^2 + \sqrt{1+8m^2}\right)^{3/2}}. \tag{62}$$

This equation is the condition that the newly formed bubble will form an inflating, semi-closed universe. The RHS of Eq. (62) is of order unity and nearly constant for $0 \leq m \leq 1$, decreasing from 0.77 at $m = 0$ to 0.59 at $m = 1$. In particular, this equation shows that such a universe will *always* form if the initial bubble radius R_0 is large enough,

$$\mu R_0 \gtrsim 0.8/\epsilon^{1/2}. \tag{63}$$

Furthermore, Eq. (62) can be shown to imply that the bubble is outside the Hubble radius when it forms, in the sense that $2M_2/R < 1$.

Equation (62) is also sufficient to form a new inflating universe even if ϵ is small and Eq. (61) is not satisfied, but it is not necessary. In that case it is easier to form a new inflating universe than implied by Eq. (62).

VI. INSTANTONS AND QUANTUM DECAY

We now wish to estimate the probability per unit time that the wall-dominated universe we have described in the previous section will decay due to quantum effects. For simplicity, we will consider the case of the VIS solution; however, in the absence of false vacuum energy, one would expect the effect of a nonvanishing Schwarzschild mass to be small. We proceed by constructing a Euclidean instanton which can be interpreted as interpolating between a time slice of the VIS solution and a time slice of a two-wall spacetime in which a second domain wall has nucleated at zero size, with the two walls tunnelling toward each other until they collide and annihilate.

We begin by calculating the action of the Euclidean VIS solution. Recall that the radius of the wall as a function of the Minkowski time in flat coordinates is given by Eq. (17):

$$X^2 + Y^2 + Z^2 - T^2 = \left(\frac{2}{\mu G}\right)^2, \tag{64}$$

and that the complete spacetime consists of the interiors of two such hyperboloids, identified at their boundaries (the wall). The Euclidean VIS solution therefore consists of 2 balls of flat 4-space, with the wall as their common (identified) boundary. Then by Eq. (8), the action of this solution is

$$\begin{aligned} I_{VIS} &= -\frac{\mu}{8\pi}\int \sqrt{h} d^3x \\ &= -\frac{\mu}{8\pi}(2\pi^2 a^3) \\ &= -\frac{2\pi}{\mu^2 G^3}. \end{aligned} \tag{65}$$

Having constructed the Euclidean VIS thin-wall solution, we can now construct a more general class of thin-wall Euclidean spacetimes, some members of which are solutions to the field equations, and therefore candidates for instantons that mediate interesting processes such as quantum tunneling. We construct these spacetimes out of some even number $2n$ ($n = 1, 2, 3\ldots$) of pieces; the Euclidean VIS solution, composed of two pieces, will be the case $n = 1$. Each piece is a lens-shaped region of flat 4-space bounded by two 3-spherical segments. The two 3-spherical segments join on a complete 2-sphere ("the edge of the lens"). We form the complete space by identifying pairwise the 3-spherical segments in round-robin fashion; all the 2-spherical edges are identified to a single 2-sphere. We demand that space be locally flat at this 2-sphere (no conical singularity — angle of 2π around any circle). In the whole space, the walls are precisely the $2n$ identified 3-spherical segments. It will now be shown that for a given value of n, a configuration which extremizes the Euclidean action consists of $2n$ identical lenses, with the two 3-spherical segments bounding each lens meeting at an angle $2\pi/2n$. Furthermore, the radius of curvature of each segment will be equal to $2/\mu G$, the radius of the VIS solution.

For a flat, compact spacetime, the action in the form of Eq. (5) becomes

$$I_{tw} = -\sum_i \int_{D_i} d^3x\sqrt{h}\left[\frac{-(K_1+K_2)+2\mu}{8\pi G}\right], \tag{66}$$

where the sum is over the $2n$ 3-spherical lens boundaries as described above. Each 3-spherical segment can be described as a segment, or cap, of S^3 of some fixed radius a_i and maximum polar angle θ_i:

$$ds^2 = a_i^2\left(d\theta^2 + \sin^2\theta d\Omega^2\right), \tag{67}$$

where $0 \le \theta \le \theta_i$.

The extrinsic curvature and its trace at the surface of a 3-sphere of radius a are given by $K_{11} = K_{22} = K_{33} = \pm 1/a$ and $K = \pm 3/a$, where the sign depends on whether the sphere is chosen to have positive or negative curvature on each side. The case of a positive-energy domain wall corresponds to choosing negative curvature on each side of the wall, *i.e.,* an observer on either side of the spherical wall is enclosed by it. Using $K_1 + K_2 = -6/a$ in the above action, we arrive at

$$I_{tw} = \frac{1}{4\pi}\sum_i \int_{D_i} d^3x\sqrt{h}\left[\frac{-3}{Ga_i}+\mu\right]. \tag{68}$$

In order that the geometry be smooth, we demand that the area of the 2-sphere which joins any pair of 3-sperical segments be the same for all pairs of segments, hence

$$A_2 = \int \sqrt{g_{\chi\chi}g_{\phi\phi}}d\chi d\phi = 4\pi a_i^2 \sin^2\theta_i = \text{const} \equiv 4\pi a^2, \tag{69}$$

where a is a constant for all lenses in a configuration. Likewise we demand smoothness (no conical singularity) at the pole of our configuration; hence

$$\sum_{i=1}^{2n}\theta_i = \pi. \tag{70}$$

These constraints can be used to eliminate the $2n$ radii a_i in favor of the single parameter a, and to eliminate one of the polar angles in favor of the remaining $2n-1$. The physical parameter space is now $(0 < \theta_i < \pi,\ a > 0)$, where $i = 1, \ldots, 2n-1$.

The action is then

$$\begin{aligned} I_n(\theta_1,\ldots,\theta_{2n-1},a) = \\ &\frac{1}{2}\sum_{i=1}^{2n-1}\left(\theta_i - \frac{\sin 2\theta_i}{2}\right)\left(\frac{\mu a^3}{\sin^3\theta_i} - \frac{3a^2}{G\sin^2\theta_i}\right) \\ &+\frac{1}{2}\left(\pi - \sum\theta_i + \frac{\sin 2\sum\theta_i}{2}\right)\left(\frac{\mu a^3}{\sin^3\sum\theta_i} - \frac{3a^2}{G\sin^2\sum\theta_i}\right). \end{aligned} \tag{71}$$

To extremize the action over our configurations, we set

$$
\begin{aligned}
0 = \frac{\partial I_n}{\partial \theta_i} &= \sin^2\theta_i \left[\mu\left(\frac{a}{\sin\theta_i}\right)^3 - \frac{3}{G}\left(\frac{a}{\sin\theta_i}\right)^2\right] \\
&+ \frac{3\cos\theta_i}{2\sin\theta_i}\left(\theta_i - \frac{\sin 2\theta_i}{2}\right)\left[\frac{2}{G}\left(\frac{a}{\sin\theta_i}\right)^2 - \mu\left(\frac{a}{\sin\theta_i}\right)^3\right] \\
&- \sin^2\Sigma\theta_i \left[\mu\left(\frac{a}{\sin\sum\theta_i}\right)^3 - \frac{3}{G}\left(\frac{a}{\sin\sum\theta_i}\right)^2\right] \\
&+ \frac{3\cos\sum\theta_i}{2\sin\sum\theta_i}\left(\pi - \sum_{i=1}^{2n-1}\theta_i - \frac{\sin 2\sum\theta_i}{2}\right) \\
&\times \left[\frac{2}{G}\left(\frac{a}{\sin\sum\theta_i}\right)^2 - \mu\left(\frac{a}{\sum\sin\theta_i}\right)^3\right],
\end{aligned} \tag{72}
$$

and

$$
\begin{aligned}
0 = \frac{\partial I_n}{\partial a} &= \frac{3}{2}\sum_{i=1}^{2n-1}\left(\frac{\theta_i - \frac{1}{2}\sin 2\theta_i}{\sin\theta_i}\right)\left[\mu\left(\frac{a}{\sin\theta_i}\right)^2 - \frac{2}{G}\left(\frac{a}{\sin\theta_i}\right)\right] \\
&+ \frac{3}{2}\left(\frac{\pi - \sum\theta_i + \frac{1}{2}\sin 2\sum\theta_i}{\sin\sum\theta_i}\right)\left[\mu\left(\frac{a}{\sin\sum\theta_i}\right)^2 - \frac{2}{G}\left(\frac{a}{\sin\sum\theta_i}\right)\right],
\end{aligned} \tag{73}
$$

which has the obvious, symmetrical solution

$$
\theta_1 = \ldots = \theta_{2n-1} = \frac{\pi}{2n}, \tag{74}
$$

$$
\frac{a}{\sin\theta_i} = \frac{2}{\mu G}. \tag{75}
$$

We believe that there are no other allowed solutions, but have not proved this. Although we have extremized the action only within our set of configurations, in fact the extremizing spacetimes are solutions of the full Euclidean field equations. Next we determine whether these extrema are minima, maxima or saddlepoints of the action.

To determine the character of an extremum of a function of several variables, one constructs the Hessian matrix, or matrix of second partial derivatives. Then the extremum is a local minimum if and only if every eigenvalue of that matrix is nonnegative, whereas each negative eigenvalue represents a direction in which the function decreases, so that if such eigenvalues exist then the extremum is a saddlepoint [13]. In the present case one finds

$$
H_{\theta_i\theta_j} = A \times \begin{pmatrix} -2 & -1 & -1 & \ldots & -1 \\ -1 & -2 & -1 & \ldots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \ldots & -1 & -2 \end{pmatrix}, \tag{76}
$$

$$
H_{aa} = B, \tag{77}
$$

$$
H_{\theta_i a} = 0, \tag{78}
$$

where A and B are positive constants for a given value of n, defined by

$$A \equiv \frac{2}{\mu^2 G^3}\left[2\sin\frac{\pi}{n} - 3\cot^2\frac{\pi}{2n}\left(\frac{\pi}{n} - \sin\frac{\pi}{n}\right)\right], \tag{79}$$

$$B \equiv \frac{3}{G}\left[\frac{\pi - n\sin\frac{\pi}{n}}{\sin^2\frac{\pi}{2n}}\right]. \tag{80}$$

The angular part of the Hessian has eigenvalues $(-2n, -1, \ldots, -1)$, so that this extremum is clearly a saddlepoint of the Euclidean action. However, since no other extrema appear to exist in the physically allowed parameter space, this symmetric solution can still be interpreted as an instanton which mediates quantum tunneling.[3]

The total Euclidean action of this solution is

$$I_n = -\frac{2\pi}{\mu^2 G^3}\left(1 - \frac{n}{\pi}\sin\frac{\pi}{n}\right). \tag{81}$$

Since this action is an increasing function of n, the VIS solution ($n = 1$) is the thin-wall solution of least action, and the next lowest lying solution will be $n = 2$, which we therefore choose as our candidate instanton. It consists of 4 lenses of flat 4-space, bounded by segments of 3-spheres which we identify pairwise and each pair of which meets at the angle $\pi/2$. One can think of this instanton as mediating the creation from nothing of two domain walls; from Eq. (69), each wall has radius of curvature $a_2 = a_1 \sin(\pi/4)$, where $a_1 = 2/(\mu G)$ is the VIS radius. Hence the two walls are created by the instanton with radius $R = \sqrt{2}/(\mu G)$.

One can choose a "final" slice of zero extrinsic curvature through this 4-geometry such that at an instant of Euclidean time, a single wall separates two domains which contain the same phase of the scalar field; the slice passes through the centers of two of the lenses as shown in Figure 3. Let us refer to these two lenses as the primary lenses. Then the slices evolve backwards in Euclidean time towards the VIS solution as follows: successive slices intersect the primary lenses in a sequence of 2-spheres of decreasing radii, with each slice passing through one of the intermediate lenses in such a way that the 2-spheres shrink faster in one of the primary lenses than in the other. This process continues until the smaller of the 2-spherical slices has reached zero radius, as shown in Figure 4; this is the point at which the second wall appears. We require that the initial slice be isomorphic to one of zero extrinsic curvature through the VIS space; however, our instanton contains no such slice.

---

[3] In general, it is not possible to determine absolutely that stable solutions to the nonlinear Eqs. (VI) do not exist in the physical parameter space ($0 < \theta_i < \pi$, $a > 0$), *i.e.,* that a local minimum of the action does not exist and make the dominant contribution to the Euclidean path integral for the tunneling amplitude. However, we have searched for such solutions numerically, and we do not believe that they exist.

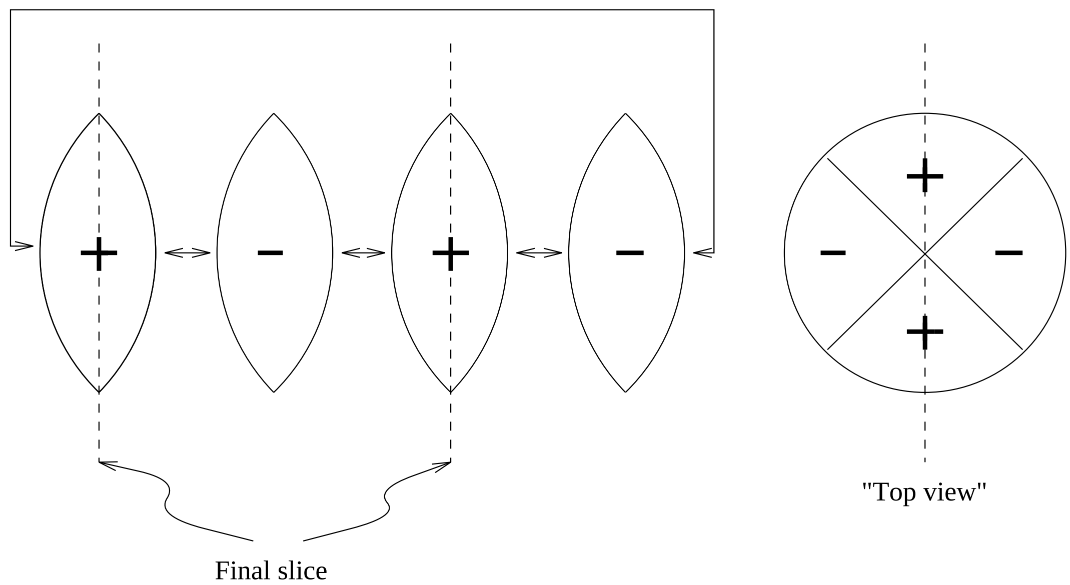

FIG. 3. A hypersurface of zero extrinsic curvature passing through the $n = 2$ symmetric instanton. The surface contains two regions of space where the phase of the scalar field is the same.

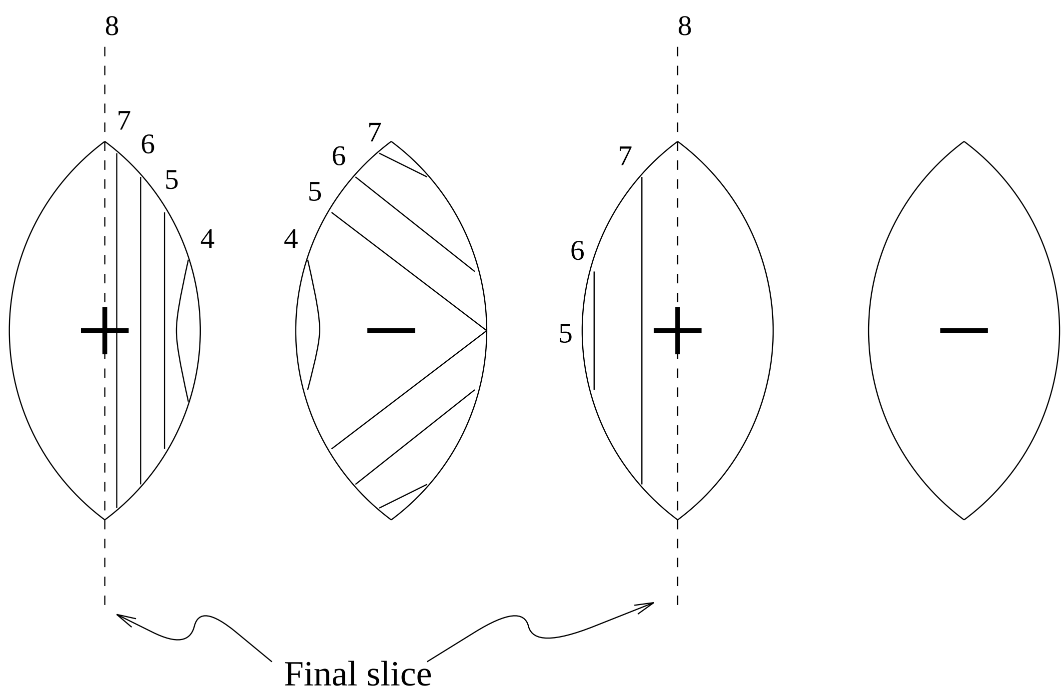

FIG. 4. The final stages of the evolution of the hypersurface which interpolates between the final and initial configurations. Successive instants of Euclidean time are numbered; slice 5 represents the instant at which the second wall is created with zero radius.

Although the meaning of this is unclear, similar pathologies have been encountered in

Euclidean quantum gravity before. One way of getting around the difficulty is simply to adopt the no-boundary proposal, *i.e.,* to calculate the barrier factors for tunneling from nothing of the initial and final field configurations, and then take their difference as the action of the interpolation. This was the approach taken, for example, in [14,15,6]. Another approach has been proposed by Farhi, Guth and Guven [16], who encountered this difficulty when calculating the amplitude for a false-vacuum bubble to tunnel through a classically forbidden region, so that it could expand perpetually. They suggested the following rule: that the Euclidean manifold exhibits a 2-sheeted structure, with the interpolating slices moving for part of their evolution on the second sheet, which would then contain a suitable initial slice. (One must then find a single slice which can be matched between the two sheets.) In their work, they called such a multi-sheeted Euclidean manifold a "pseudomanifold", [4] and they defined the covering number of any point in the manifold as the number of times the point is crossed by the evolving hypersurface in the future direction, minus the number of times it is crossed in the past direction. They then found that the action weighted by the covering number yields the correct Euclidean equations of motion.

In the present case the second sheet, on which the slices evolve before the new domain wall has appeared, is simply the Euclidean VIS manifold. The evolution then proceeds as shown in Figure 5. The action of the interpolation is calculated by following the hypersurface through its complete evolution, and summing the wall area (and thus the action) "swept out" by it, weighted by the covering number as described above. For the first part of the evolution (slices 1–4 in Fig. 5), the action is

$$\frac{1}{2}I_{VIS} - \Delta I, \tag{82}$$

where I_{VIS} is the Euclidean action of the VIS solution, and ΔI is the action of the area of the 3-hemisphere *not* swept out by this part of the evolution. Now the slice begins evolving on that part of the manifold containing the instanton, and the action of this part of the interpolation is

$$-\left(\frac{1}{2}I_2 - \Delta I\right), \tag{83}$$

where I_2 is the Euclidean action of the instanton, and the minus sign is present because the slice evolves through these points with the opposite orientation. Summing the two contributions, we see that the total Euclidean action of the interpolation is

$$I = \frac{1}{2}(I_{VIS} - I_2). \tag{84}$$

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[4]Not to be confused with a pseudomanifold in topology.

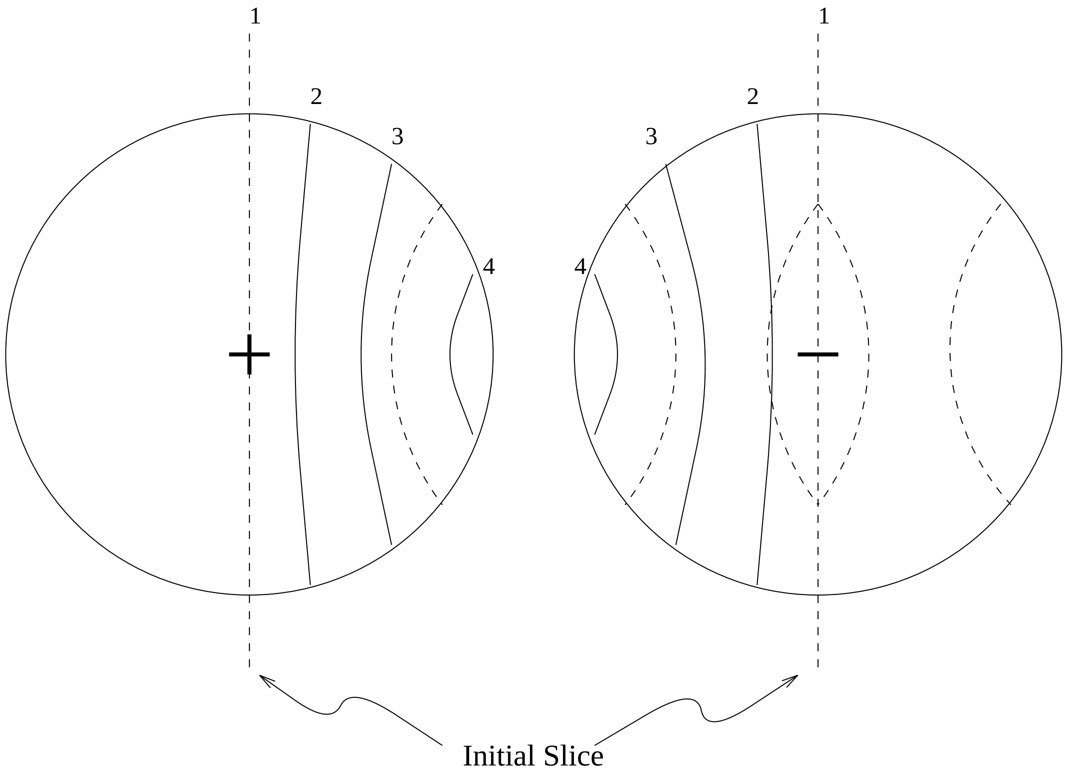

FIG. 5. The initial stages of the evolution of the interpolating hypersurface, which take place on the VIS sheet of the manifold. The second sheet, consisting of the $n = 2$ instanton, is indicated by the curved, dashed lines.

A physical interpretation of this process is that in one half of the VIS space, a second wall is nucleated quantum mechanically and grows in imaginary time while the first wall shrinks; when the walls reach the same place and same size, they annihilate in a burst of energy. In other words, one imagines the scalar field in one half of the space fluctuating, in its entirety, into the opposite vacuum state. At the same time, the geometry fluctuates so that the domain wall radius is reduced by a factor of $\sqrt{2}$ at the instant of its decay.

The Euclidean action of the instanton is

$$\begin{aligned} I_2 &= \frac{-2\pi}{\mu^2 G^3}\left(1 - \frac{2}{\pi}\sin\frac{\pi}{2}\right) \\ &= \frac{4-2\pi}{\mu^2 G^3}. \end{aligned} \tag{85}$$

The action of the interpolation is therefore

$$I = \frac{1}{2}(I_{VIS} - I_2) = -\frac{2}{\mu^2 G^3}. \tag{86}$$

The amplitude for the decay process is then given in the semiclassical limit by [17]

$$\frac{\Gamma}{V} = A \exp[-I], \tag{87}$$

where the prefactor A can by calculated by considering perturbations about the instanton [8]. Ignoring the prefactor, we thus find that the probability for decay will contain the barrier factor

$$P \sim \exp\left[-\frac{2}{\mu^2 G^3}\right]. \tag{88}$$

VII. CONCLUSIONS

The study of domain walls in the early universe is motivated both by their inflationary nature, and by the naturalness with which they arise from fundamental field theories. The spacetimes associated with domain walls are in many ways analagous to deSitter spacetime, with the vacuum energy confined to the two-dimensional sheets which are the walls. However, the presence of domain walls does not require a global source of vacuum energy; only a broken discrete symmetry need exist to lead to the formation of the walls. Domain walls may arise more naturally in microphysics than the false vacuum which commonly drives models of inflation.

As in the case of false vacuum energy, domain walls tend to dominate the classical dynamics of the universe due to their gravitational properties. However, the resulting cosmological models exhibit large-scale anisotropies in the CMBR, inconsistent with observations, and although the instability of the false vacuum is well-known [17], there was until now no corresponding result in the case of pure domain walls. For this reason, it has long been believed that domain walls should be forbidden in the early universe. The primary result of this paper has been to demonstrate that closed domain walls are in fact quantum mechanically unstable, and will therefore decay with finite probability.

Here we have focused on the evolution of closed, spherically symmetric domain walls, which in general have a Schwarzschild mass and an associated singularity. In our analysis of the classical dynamics of these spacetimes, we have shown that four general classes of behavior are possible: a wall may be born with zero size, expand to a finite maximum radius and recollapse; it may be born with zero size and expand indefinitely; it may collapse from infinite size to a minimum radius and then reexpand; or it may collapse from infinite size to zero size. In the case where the domain wall expands indefinitely, it passes inside the black hole horizon, avoids the singularity and creates a new, inflating universe which is causally disconnected from the original spacetime, as first pointed out by Blau et al. [2]. We have shown that this process will occur naturally if the spherical domain wall is formed with radius larger than the Hubble radius at formation.

Our toy model to study the decay of such a domain wall-dominated universe was chosen to be the limiting case where the Schwarzschild mass of the wall vanishes. In this case, we have seen that there is an instanton which mediates the decay process, and that the decay probability per unit time is

$$P \sim \exp\left[-\frac{2}{\mu^2 G^3}\right]. \tag{89}$$

Although the process is heavily suppressed, we emphasize that in our scenario there are no competing processes; the domain wall simply expands until it decays, whenever that may be. The energy of the wall will subsequently become thermalized and lead to a hot big bang.

Our instanton calculation also predicts the classical state immediately after the decay, namely a closed universe of smaller volume, consisting of two regions of flat space meeting at a wall of energy. The wall of energy is the annihilation product of the two domain walls; its geometry is that of a 2-sphere with a radius $\sqrt{2}/\mu G$, or a factor $1/\sqrt{2}$ smaller than the radius of the domain wall before the decay.

Discussion of the subsequent fate of this new universe is beyond the scope of this paper. But this closed universe now contains a form of energy which obeys all the usual energy conditions, and therefore recollapse may be expected. If the wall of energy is in a metastable state ("slow rolldown") then recollapse may be delayed greatly, however, and this universe might expand to a much greater radius before recollapse.

Our instanton calculation has also raised some interesting questions about the overall nature and validity of the instanton approach to quantum gravity. In general, one would like to find a Euclidean instanton which, in addition to being the least-action solution of the Euclidean field equations, satisfies two conditions: (i) it contains slices which are isomorphic to static slices in both the final and the initial state of the tunneling process, and (ii) the hypersurface which interpolates between these two slices has a unique trajectory. However, as in our calculation, one finds that this is not always possible; in such cases, it is not clear how to proceed. We have followed the rule of Farhi, Guth and Guven [16], to evolve the interpolating hypersurface on a 2-sheeted "pseudomanifold" made of two instantons glued together over a common region, wherein the first sheet contains the final slice, and the second sheet which contains the initial slice. The two parts of the evolution are joined on a surface in the common region. An alternate approach, which has become standard practice (see, for example, [14,6]), is simply to subtract the action of the instanton which mediates "tunneling from nothing" of the initial state, from that of the instanton which describes "tunneling from nothing" of the final state, and take this result as the action of the interpolation. This approach does not even require that the two instantons have a single hypersurface which can be identified. The two approaches give identical answers in the present case, although the reasons for this are unclear.

Furthermore, our instanton is a saddlepoint, rather than a minimum, of the Euclidean action for the class of field configurations we have considered, as invariably happens in quantum gravity. Since there do not appear to be any extrema of lesser action, we have taken our solution to be the dominant contribution to the decay process. However, we have not proven that this is the case.

In order to avoid the ambiguities in the instanton calculation, we turn directly to the canonical quantization of the the minisuperspace model corresponding to the domain wall spacetime under consideration. Since the physical interpretation of the decay process is that of a second domain wall being nucleated in one half of the existing spacetime, and the two walls tunneling towards each other, there will be two degrees of freedom in the model. We will carry out the canonical quantization of such a two-wall spacetime in the next paper [18], and we will find a significantly different result for the quantum decay.

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